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NESTED BALANCED TERNARY, QUATERNARY AND PARTIALLY BALANCED TERNARY RECTANGULAR DESIGNS AND THEIR APPLICATIONS

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Abstract

This paper is concerned with the recursive construction of nested balanced ternary (NBT), nested balanced quaternary (NBQ) and partially balanced ternary rectangular (PBTR) designs through a set of balanced incomplete block (BIB) designs. An illustrative example in each case has been added separately. The efficiency of NBT and NBQ designs has also been computed.

Key Words: Balanced Incomplete Block (BIB) Design; Incidence Matrix; Nested Balanced Ternary (NBT) Design, Nested Balanced Quaternary (NBQ) Design, Partially Balanced Ternary Rectangular (PBTR) Design.

1. Introduction

The heterogeneity of experimental material should be taken care of critically in designing of an experiment; otherwise the real treatment differences remain undetected, unless they are large enough. The blocking is a kind of technique which is used to bring about homogeneity of experimental units within block for the case of one way elimination of heterogeneity, so that treatment contrasts can be estimated. Due to the presence of multiple factors of heterogeneity, blocking alone in one direction may fail to remove such heterogeneity.

Preece (1967) introduced a class of designs, called nested balanced incomplete block designs, in which ,within each block there is one nuisance factor nested. This idea has been generalized to nested partially balanced incomplete block deigns by Homel and Robinson (1975). The various constructions of such nested designs have been considered in Dey (1986), and Banerjee and Kageyama(1990). It is to be noted that in the nested designs constructed in the literature, the sub blocks and super –blocks have the same association scheme.

Vartak (1955) introduced the concept of rectangular designs which are 3 – associate partially balanced incomplete block(PBIB) designs based on a rectangular association scheme of v=mn treatments arranged in an mxn rectangle such that with respect to each treatment, the first associates are the other n-1 (n_1, say) treatments of the same row, the second associates are the other m-1 (=n_2 say) treatments of the same columns and the remaining (m-1)(n-1) (=n_3 say treatments of the 3^{rd} associates).

Sharma and Tiwari (2011) have constructed of a series of partially balanced ternary rectangular designs giving some examples in the range of $R \leq 50$ and $K \leq 19$.

The objective of this paper is to develop recursive methods for the construction of nested balanced ternary (NBT), nested balanced quaternary (NBQ) and partially balanced ternary rectangular (PBTR) designs through a set of balanced incomplete block (BIB) designs. An illustrative example in each case has been added separately. The efficiency of NBT and NBQ designs has also been computed.

2. Definitions and Notations

2.1 Nested Balanced Incomplete Block Design

A nested BIB design with parameters v, b_1 ,r ,k_1, b_2, k_2, λ_1, λ_2 is a block design in which both the nesting blocks (superblocks) and the sub-blocks form BIBD (v, b_1 , r, k_1, λ_1) and BIBD (v, b_2, r, k_2 ,λ_2), respectively (see Preece , 1967 ; Morgan ,1996). Thus, there are b_2 sub – blocks of size k_2 nested in each of b_1 superblocks of size k_1. It is to be noted that parameters λ_1 and λ_2 in the nested BIB design have different meaning from those in partially balanced incomplete block (PBIB) designs.

2.2 Nested Balanced Ternary Design

A balanced n-ary design where n=3 is known as the balanced ternary design (BTD) which is a collection of B blocks, each of cardinality K ($K \leq V$), chosen from a set of size V in such a way that each of the V treatments occurs R times altogether, each of the treatments occurring once in a precisely Q_1 blocks and twice in precisely Q_2 blocks, and with incidence matrix having inner product of any two rows Λ is denoted by BTD (V,B,Q_1,Q_2,R,K,Λ) such that the second system is nested within the first with each block from the first system then the nested balanced ternary design is prepared by addition of the blocks of the second system generated from the first system.

2.3 Nested Balanced Quaternary Design

A balanced n-ary design where n=4 is known as the quaternary design which is a collection of B blocks, each of cardinality K ($K \leq V$), chosen from a set of size V in such a way that each of the V treatments occurs R times altogether, each of the treatments occurring once in a precisely Q_1 blocks, twice in precisely Q_2 blocks and thrice in precisely Q_3 blocks, and with incidence matrix having inner product of any two rows Λ is denoted by balanced quaternary design (BQD) (V,B,Q_1,Q_2,Q_3,R,K,Λ) such that the third system is nested within the second with each block, and second system is nested within the first with each block from the first system then the nested balanced quaternary design is developed by addition of the blocks of the third system generated from the second system, and second system generated from the first system.

2.4 Rectangular Design

A rectangular design is an arrangement of v = mn treatments in b blocks such that

I. Each block contains k distinct treatments , $k < v$
II. Each treatment occurs in exactly r blocks

III. The mn treatments are arranged in a rectangle of m rows and n columns such that any two treatments in the same row (column) occur together in λ_1,λ_2 blocks respectively, and λ_3 blocks otherwise.

It is known (Vartak, 1955) that the Kronecker product of incidence matrices of two BIB designs produces a rectangular design.

The rectangular association is here arranged as

1	2	… n
n + 1	n + 2	2n
(m-1)n+1	(m-1)n+2 …	mn

Here , $v = mn$, $n_1 = n-1$, $n_2 = m-1$, $n_3 = (m-1)(n-1)$.

2.5 Partially Balanced Ternary Rectangular (PBTR) design

Let A be a set of V treatments arranged in B blocks. Let $\{G_i / i = 1,2,\ldots,m\}$ be a partition of A into m sets each of size n, called groups. The groups define an association scheme on A with two classes ;two treatments are first associates if they belong to the same group and are second and third associates otherwise. A PBRTD with parameters $(m,n,V,B,Q_1,Q_2,R,K,\Lambda_1,\Lambda_2,\Lambda_3)$ is defined to be an incidence structure satisfying

$\sum_{j=1}^{B} n_{ij} = R$ for each $i = 1,2,\ldots,V$
$\sum_{i=1}^{V} n_{ij} = K$ for each $j = 1,2,\ldots,B$

and $\sum_{k=1}^{B} n_{ik}n_{jk} = \Lambda_1$ if treatments i and j $(i \neq j)$ are first associates
$= \Lambda_2$ if treatments i and j $(i \neq j)$ are second associates
$= \Lambda_3$ if treatments i and j $(i \neq j)$ are third associates

where n_{ij} = number of times i^{th} treatment occurs in j^{th} block , $n_{ij} \in \{0,1,2\}$
$i = 1,2,\ldots,V$; $j = 1,2,\ldots,B$

Each treatment occurs with multiplicity ‘1’ in Q_1 blocks and with multiplicity ‘2’ in Q_2 blocks.

For a PBTR design following two results hold.

$VR= BK$

$\Lambda_1 (n-1) + \Lambda_2 (m-1) + \Lambda_3 (mn-n-m+1) = R (K-1) – 2Q_2$

3. Construction

Theorem 3.1: The existence of BIB design with parameter v, b, r, k, λ implies the existence of nested balanced ternary design with following parameters: V = v, B = bk, Q_1 = r, Q_2 = r(k-1), R = r(2k-1),K = 2k-1, Λ = 4λ (k-2) + 4λ.

Proof

It is obvious to have the same number of treatments as in the design

Therefore, V = v.

B : The number of blocks in the design is certainly equal to the multiplication of the number of blocks in initial design and the number of blocks which is nested by taking $\binom{k}{k-1}$ i.e. is equal to bk.

Therefore, B = bk.

Q_1 : The multiplicity of treatment '1' in a block is obtained by the multiplication of r and b-r from the nested design. Since, the nested design at the 2nd stage becomes BIBD consisting of the parameters v=b= k(equal to block size of the first system), r=k= k-1 (equal to the replication of the first system), λ= k-2,

Therefore, Q_1 = r.

Q_2 : The multiplicity of treatments '2' in a block is obtained by the multiplication of r (number of replications) from the 1st stage and the replication number which is being nested at the 2nd stage.

Therefore, Q_2 = r(k-1).

R : The total number of replications is equal to $Q_1 + 2Q_2$.

Therefore, R= r(2k-1).

K : The block size of the nested design is equal to the k + k-1 = 2k-1 i.e the addition of the block size of the 1st stage and the 2nd stage.

Therefore, K = 2k-1.

Λ : In order to find the value of Λ in the nested BTD will occur as (2,2),(2,1),(1,2).

For the ordered pair (2,2), the λ at the 1st stage is multiplied by λ at the 2nd stage i.e.(k-2).

Thus,it is equal to $4\,\lambda(k-2)$.

For the ordered pair (2,1) and (1,2), it is equal to the multiplication of λ at the 1st stage and r- λ at the 2nd stage which is equal to λ(r- λ).Since, r- λ in the nested design is equal to 1. Thus, it is equal to 4λ.

Thus, the total Λ = 4λ (k-2) + 4λ.

Hence Q.E.D..

Example 3.1.1: Let us consider BIB design with parameters v = 3, b = 3, r = 2, k = 2, λ = 1 . Applying Theorem 3.1, it is developed as a nested balanced ternary design (Table 3.1.1)

B_1	B_2	B_3	B_4	B_5	B_6
1	1	1	1	2	2
1	3	1	2	2	3
3	3	2	2	3	3

Table 3.1.1: The number of blocks in NBT design with parameters V = 3, B = 6, Q_1 = 2, Q_2 = 2, R = 6, K = 3, Λ=4

Efficiency of this design is 0.666

Example 3.1.2 Let us consider doubly BIB design with parameters v = 4, b = 6, r = 3, k = 2, λ = 1 which is developed as a nested balanced ternary design with V = 4, B = 12, Q_1 = 3, Q_2 = 3, R = 9, K = 3, Λ = 4.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
1	1	2	2	3	3	2	2	1	1	1	1
1	4	2	4	3	4	2	3	1	3	1	2
4	4	4	4	4	4	3	3	3	3	2	2

Table 3.1.2: The number of blocks in NBT design with parameters V = 4, B = 12, Q_1 = 3, Q_2 = 3, R = 9, K = 3, Λ = 4

Efficiency of this design is 0.592.

Example 3.1.3: Let us consider BIB design with parameters v = b =7, r = k= 3, λ = 1 which is developed as a nested balanced ternary design with V = 7, B = 21, Q_1 = 3, Q_2 = 6, R = 15, K = 5, Λ = 8

Efficiency of this design is 0.746.

Theorem 3.2: The existence of BIB design with parameter v, b, r, k, λ implies the existence of nested balanced quaternary (NBQ)design with following parameters: V = v, B = bk(k-1), Q_1 = r(k-1), Q_2 = r(k-1), Q_3 = r(k-1)(k-2),R =3r(k^2-2r+1), K = 3(k-1), $\Lambda = 9\lambda(k-2)(k-3) + 12\lambda(k-2) + 6\lambda + 4\lambda$.

Proof It is obvious to have the same number of treatments as in the design

Therefore, V = v.

B : It is obtained by the multiplication of the number of blocks nested at the three stages.

Therefore, B = bk (k-1).

Q_1 : The multiplicity of the treatment '1' is obtained by the multiplication of the replication number r at the 1st stage and the replication number nested at the 2nd stage i.e.(k-1).

Therefore, $Q_1 = r(k-1)$.

Q_2 : The multiplicity of treatment '2' is obtained by the multiplication of the replication number r at the 1st stage and the multiplication of the replication (k-1) from the nested design.

Therefore, $Q_2 = r(k-1)$.

Q_3 : The multiplicity of treatment '3' in the design is obtained by the replication number generated at the 3rd stage which is equal to r(k-1)(k-2).

Therefore, $Q_3 = r(k-1)(k-2)$.

R : The total replication is equal to $Q_1 + 2Q_2 + 3Q_3$.

Therefore, $R= 3r(k^2-2r+1)$.

K : The block size of the nested is equal to the k+(k-1)+(k-2) i.e the addition of the block size of the 1st stage, 2nd stage and 3rd stage.

Therefore, $K = 3(k-1)$.

Λ : In order to find the value of Λ in the nested BQD will occur as (3,3),(3,2),(2,3),(1,3),(3,1),(2,2),(1,2),(2,1).

For the ordered pair (3,3), it is obtained by multiplication of λ's at the three stages i.e.$\lambda(k-2)(k-3)$.

For the ordered pair (3,2) or (2,3), it is obtained by multiplication at the three stages

i.e. $\lambda(k-2)(1)$ [the value of (1,0) in 3rd stage is equal to 1].

For the ordered pair (3,1) or (1,3), it is obtained by multiplication at the three stages i.e. $\lambda(1)(1) = \lambda$.

For the ordered pair (2,2), it is equal to λ's and number of occurrence of λ at the 3rd stage i.e. $\lambda(k-2)(0) = 0$.

For the ordered pair (2,1) or (1,2) occurrence, it is obtained by multiplication of λ by 2 stages only i.e. λ.

Thus, the total $\Lambda = 9\lambda(k-2)(k-3) + 12\lambda(k-2) + 6\lambda + 4\lambda$.

Hence, Q.E.D.

Example 3.2.1: Let us consider BIB design with parameters v = b = 4, r = k= 3, $\lambda = 2$. Applying Theorem 3.2 which is developed as a nested balanced quaternary design.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
1	1	1	1	3	3	1	1	1	1	2	2	1	1	1	1	2	2	2	2	2	2	3	3
1	3	1	3	3	3	1	2	1	2	2	2	1	2	1	2	2	2	2	3	2	3	3	3
3	3	3	4	3	4	2	2	2	4	2	4	2	2	2	3	2	3	3	3	3	4	3	4
3	3	4	4	4	4	2	2	4	4	4	4	2	2	3	3	3	3	3	3	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	4	4	4	4	4	4

Table 3.2.1 The number of blocks with parameters V = 4, B = 24, $Q_1 = 6$, $Q_2 = 6$, $Q_3 = 6$, R = 36, K = 6, $\Lambda = 44$

Efficiency of this design is 0.814.

Example 3.2.2: Let us consider BIB design with parameters v = 7, b = 7, r = 3, k = 3, $\lambda = 1$ which is developed as a nested balanced quaternary design with parameters V = 7, B = 42, $Q_1 = 6$, $Q_2 = 6$, $Q_3 = 6$, R = 36, K = 6, $\Lambda = 22$.

Efficiency of this design is 0.712.

Theorem 3.3: The existence of rectangular design with parameters v = mn, b, r, k, λ_1, λ_2, λ_3 implies the existence of a partially balanced ternary rectangular (PBTR) design with parameters V = v = mn, B = b $\left[\binom{n'_1}{2} + \binom{n'_2}{2}\right]$, $Q_1 = r\left[\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2}\right]$, $Q_2 = r\,[(n'_1 - 1)]$, $R = Q_1 + 2Q_2$, K = k+2, $\Lambda_1 = 4\lambda_1 + 4\lambda_1(n'_1 - 2) + \lambda_1\left[\binom{n'_1 - 2}{2} + \binom{n'_2}{2}\right]$, $\Lambda_2 = 4\lambda_2(n'_1 - 1) + \lambda_2\left[\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2} - (n'_2 - 1)\right]$, $\Lambda_3 = 4\lambda_3(n'_1 - 1) + \lambda_3\left[\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2} - (n'_2 - 1)\right]$.

where, n'_1 and n'_2 are the number of treatments which form the block size k in rectangular designs taking as first row, second row, or first row , third row or subsequently the combination of all rows with the first row.

Proof: Here, PBTRD is constructed by the addition of pairs of first associates treatments, which occur in those blocks assuming that number of treatments in first associates is less rather than that of second associate of treatments.

The parameters are V and K are obviously explainable.

The other parameters of PBTRD are explained below:

B = By adding the pairs of first associate treatments which are in a block of size k once at a time is given by $\binom{n'_1}{2}$ and $\binom{n'_2}{2}$ because n'_1 and n'_2 are lying in the first row and second row. Thus, the total number of pairs of first associate treatments will be $\binom{n'_1}{2} + \binom{n'_2}{2}$, and then out of b blocks, the total number of blocks will be b $\left[\binom{n'_1}{2} + \binom{n'_2}{2}\right]$.

Q_1 = In PBTRD, since adding pairs of first associate treatments, the multiplicity of treatment '1' will be $\binom{n'_1}{2} - (n'_1 - 1)$ from n'_1 number of treatments and the multiplicity of '1' occur for the combination of $\binom{n'_2}{2}$.

Therefore, the multiplicity of one treatment Q_1 becomes $\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2}$ and for r number of replications, then, it becomes, r $[\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2}]$.

Q_2 = In order to get the multiplicity '2' of a treatment in a design after adding the pairs of first associates treatments , it will be $(n'_1 - 1)$ for one replication. Hence for r number of replications, it is equal to r $[(n'_1 - 1)]$.

$R = Q_1 + 2\, Q_2$.

Λ_1 = let us consider a treatment pair (x,y) whereas x and y are first associates.To each of the λ_1 blocks of PBTR design containing treatment pair (x,y). If pair (x,y) is added , we obtained λ_1 blocks in which the treatment pair (x,y) occurs four times and the first term in the expression of Λ_1 is $4\lambda_1$.

There are $(n'_1 - 2)$ pairs of first associate treatment other than treatment pair (x,y) which contain treatment x. If these $(n'_1 - 2)$ pairs are added to each of the λ_1 blocks, then we get $\lambda_1(n'_1 - 2)$ blocks in which treatment x appears with multiplicity'2' y occurs with multiplicity '1' i.e. treatment pairs (x,y) occurs twice. In addition, by adding $(n'_1 - 2)$ pairs of first associate treatment which contains treatment y other than the pair (x,y) to each of the λ_1 blocks, we get $\lambda_1(n'_1 - 2)$ blocks in which treatment pair (x,y) occurs twice. Thus, second term in the expression of Λ_1 is $4\,\lambda_1(n'_1 - 2)$.

There are $[\binom{n'_1 - 2}{2} + \binom{n'_2}{2}]$ pairs (x,y) of first associates treatments other than treatment pairs (x,y). If these pairs of treatments are added $2\lambda_1$ blocks then we get λ_1 $[\binom{n'_1 - 2}{2} + \binom{n'_2}{2}]$ blocks in which the treatment pairs (x,y) occurs with multiplicity '1'. Hence, the third term expression in the expression of Λ_1 is $\binom{n'_1 - 2}{2} + \binom{n'_2}{2}$. Therefore, $\Lambda_1 = 4\lambda_1 + 4\lambda_1(n'_1 - 2) + \lambda_1 [\binom{n'_1 - 2}{2} + \binom{n'_2}{2}]$.

Λ_2 and Λ_3: Since adding the pairs of first associate of treatments ,then the treatment pair occurs with multiplicity '2' and '1' and vice versa.

Therefore, for ordered pair (2,1),(1,2), the first terms in Λ_2 and Λ_3 is $4\lambda_1(n'_1 - 1)$.

Since, one treatment is less for the Λ_2 and Λ_3.

For ordered pair (1,1) we have the value of Λ_2 from n'_1 and n'_2 which will be $\binom{n'_1}{2}$ - $(n'_1 - 1)$ from the first part + $\binom{n'_2}{2}$ - $(n'_2 - 1)$ from the second part and have the second term.

On this similar basis of Λ_2, we can get Λ_3 replacing Λ_2 by Λ_3 only.

Hence,

$$\Lambda_1 = 4\lambda_1 + 4\lambda_1(n'_1 - 2) + \lambda_1 \left[\binom{n'_1 - 2}{2} + \binom{n'_2}{2} \right]$$

$$\Lambda_2 = 4\lambda_2(n'_1 - 1) + \lambda_2 \left[\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2} - (n'_2 - 1) \right]$$

$$\Lambda_3 = 4\lambda_3(n'_1 - 1) + \lambda_3 \left[\binom{n'_1}{2} - (n'_1 - 1) + \binom{n'_2}{2} - (n'_2 - 1) \right]$$

Hence Q.E.D.

Example 3.3.1: Let us consider rectangular BIB design with parameters V = 6, B = 3, r = 2, k = 4, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 1$, m = 2, n = 3 with the association scheme

1 2 3

4 5 6

Applying Theorem 3.3 which is developed as a PBTR design with parameters V = 6, B = 6, $Q_1 = 2$, $Q_2 = 2$, R =6, K =6, $\Lambda_1 = 5$, $\Lambda_2 = 8$, $\Lambda_3 = 4$, m = 2, n = 3.

B_1	B_2	B_3	B_4	B_5	B_6
1	1	1	1	2	2
1	2	1	3	2	3
2	4	3	4	3	5
2	4	3	4	3	5
4	5	4	6	5	6
5	5	4	6	6	6

Table 3.3.1: Number of blocks for PBTR design with parameters V = 6, B = 6, Q_1 =2, Q_2 = 2, R =6, K =6, Λ_1 = 5, Λ_2 = 8, Λ_3 = 4, m = 2, n = 3

Theorem 3.4: The existence of rectangular designs with the parameters v = mn, b, r, k, λ_1, λ_2, λ_3 implies the existence of PBTR design with the parameters : V = mn, B = b(b-1)/2, Q_1 = r(b-r), Q_2 = r(r-1)/2, R = r(b-r), K = 2k, $\Lambda_1 = r^2 + \lambda_1(b-2)$, $\Lambda_2 = r^2 + \lambda_2(b-2)$, $\Lambda_3 = r^2 + \lambda_3(b-2)$.

Proof: Here , PBTR design are constructed by taking the combination of two blocks of the existing rectangular design together at a time.

The number of treatments is obviously V = mn.

The total number of blocks is B = b(b-1)/2 since two blocks are together taken at a time. Thus,the parameters V,B, and K need no explanation.

Remaining parameters are explained below:

Q_1 : Let us consider a block containing a particular treatment x. This will occur by taking the combination of r and b-r which is equal to

$Q_1 = \binom{r}{1} \times \binom{b-r}{1} = r(b-r)$

Q_2 : This will occur by taking the combination of two treatments out of r and 0 combination out of b-r which is equal to

$Q_2 = \binom{r}{2} = r(r-1)/2$

R : Replication number R for treatment x is $R = Q_1 + 2Q_.$

Hence, R = r(b-r)

K : 2k

Λ : This parameter will consist of (2,2),(2,1),(1,2) and (1,1) ordered pairs of treatments.

For ordered pair (2,2), we consider 2's of the total λ's. Therefore , it is equal to $\binom{\lambda}{2}$.

For ordered pair (2,1) and (1,2), we consider the total out of λ's, we take 1 and 1 from (r-λ) and λ. Therefore , it is equal to $\binom{\lambda}{1}\binom{r-\lambda}{1}$.

For ordered pair (1,1),we consider 1 λ's and one combination of (0,0) and combination of (1,0) and (0,1) which is equal to $\binom{\lambda}{1}\binom{b-2r+\lambda}{1} + \binom{r-\lambda}{1}\binom{r-\lambda}{1}$.

Thus,

$$\Lambda = r^2 + \lambda\,(b-2)$$

Similarly,

$$\Lambda_1 = r^2 + \lambda_1\,(b-2)$$

$$\Lambda_2 = r^2 + \lambda_2\,(b-2)$$

$$\Lambda_3 = r^2 + \lambda_3\,(b-2)$$

$\Lambda_1(n-1) + \Lambda_2(m-1) + \Lambda_3(mn-n-m+1) = R(K-1) - 2Q_2$.

Hence Q.E.D.

Example 3.4.1: Let us consider rectangular BIB design with parameters V = 6, B = 6, r = 3, k = 3, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 1$, m = 2, n = 3 with the association scheme

1 2 3

4 5 6

Applying Theorem 3.4 which is developed as a PBTR design (given in Table 3.4.1)

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	1	1	1	1	1	1	1	2	2	1	1	2
1	2	1	2	3	2	1	2	2	2	2	3	2	3	3
2	3	3	4	4	2	2	2	3	3	3	3	3	3	4
4	4	4	4	5	3	3	4	5	3	4	5	4	4	5
5	6	4	5	6	5	4	5	5	4	5	6	5	5	5
6	6	6	6	6	6	5	5	6	6	6	6	5	6	6

Table 3.4.1 The number of blocks in PBTR deign with parameters V = 6, B = 15, $Q_1 = 9$, $Q_2 = 3$, R =15, K = 6, $\Lambda_1 = 13$, $\Lambda_2 = 17$, $\Lambda_3 = 13$, m = 2, n = 3

Theorem 3.5: The existence of rectangular designs with the parameters v = mn, b, r, k, λ_1, λ_2, λ_3 implies the existence of partially balanced quaternary rectangular(PBQR) design with the parameters :

V = mn, B=b(b-1)(b-2)/6, Q_1 = r (b-r)(b-r-1)/2, Q_2 = r (r-1)(b-r)/2, Q_3 = r (r-1)(r-2)/6, R = rb^2-3br+2r/2, K = 3k, $\Lambda_1 = 2\lambda_1^2 + 6\lambda_1 - 4b\lambda_1 + 28r^2\lambda_1 - 4r^2 + \lambda b^2 - b\lambda_1 + 2r^2b/2$.

$\Lambda_2 = 2\lambda_2^2 + 6\lambda_2 - 4b\lambda_2 + 28r^2\lambda_2 - 4r^2 + \lambda b^2 - b\lambda_2 + 2r^2b/2$.

$\Lambda_3 = 2\lambda_3^2 + 6\lambda_3 - 4b\lambda_3 + 28r^2\lambda_3 - 4r^2 + \lambda b^2 - b\lambda_3 + 2r^2b/2$.

Proof: With the existing rectangular design, the number of treatments is obviously V = mn.

The total number of blocks is B $=\frac{b(b-1)(b-2)}{6}$.

The parameters *V,B,* and *K* need no explanation.

Remaining parameters are explained below:

Q_1 : Let us consider a block containing a particular treatment *x*. This will occur by taking the combination of *r* and *b-r* which is equal to

$$Q_1=\binom{r}{1}\times\binom{b-r}{2} = \frac{r(b-r)b-r-1)}{2}$$

Q_2 : This will occur by taking the combination ot two treatments i.e. *r* and *b-r* which is equal to

$$Q_2=\binom{r}{2}\times\binom{b-r}{1} = \frac{r(r-1)(b-r)}{2}$$

$$Q_3 =\binom{r}{3} = \frac{r(r-1)(r-2)}{6}$$

R : Replication number *R* for treatment *x* is $R = Q_1 + 2Q_2 + 3Q_3$.

Hence, R = rb^2-3br+2r/2

K : 3k

Λ : This parameter will consist of

(3,3),(3,2),(2,3),(3,1),(1,3),(2,2),(2,1),(1,2),(1,1) ordered pairs of treatments.

For ordered pair (3,3), we consider 3's of the total λ's.Therefore , it is equal to $\binom{\lambda}{3}$.

For ordered pair (3,2) and (2,3), we consider the 2's of λ's and one of (r-λ) which is equal to $\binom{\lambda}{2} \times \binom{r-\lambda}{1}$

For ordered pair (3,1) and (1,3), we consider the 1's of λ's and 2's of (r-λ) which is equal to $\binom{\lambda}{1} \times \binom{r-\lambda}{2}$

For ordered pair (2,2),we consider 2's of λ's and one of (0,0) and combination of (1,1,0) and (0,1,1) , we consider one of λ's and 2's of (r-λ) which is equal to

$$\binom{\lambda}{2}\binom{b-2r+\lambda}{1} + \binom{\lambda}{1}\binom{r-\lambda}{1}\binom{r-\lambda}{1}$$

For ordered pair (2,1) and (1,2), we consider the one of λ's and one of the (r-λ) and (b-2r+λ) and other combination of (1,0,0) and (0,1,0), we consider 2's and one of (r-λ) which is equal to $\binom{\lambda}{1}\binom{r-\lambda}{1}\binom{b-2r+\lambda}{1}+\binom{r-\lambda}{2}\binom{r-\lambda}{1}$.

For ordered pair (1,1), we consider one of λ's and 2's of (b-2r+λ) and other combination of (1,0,0) and (0,1,0), we consider of 2's of (r-λ) and one of (b-2r+λ) which is equal to $\binom{\lambda}{1}\binom{b-2r+\lambda}{2}+\binom{r-\lambda}{1}\binom{r-\lambda}{1}\binom{b-2r+\lambda}{1}$

Thus, $\Lambda = 9\binom{\lambda}{3} + 12\left[\binom{\lambda}{2}\binom{r-\lambda}{1}\right] + 6\left[\binom{\lambda}{1}\binom{r-\lambda}{2}\right] + 4\left[\binom{\lambda}{2}\binom{b-2r+\lambda}{1} + \binom{\lambda}{1}\binom{r-\lambda}{1}\binom{r-\lambda}{1}\right] + 4\left[\binom{\lambda}{51}\binom{r-\lambda}{1}\binom{b-2r+\lambda}{1} + \binom{r-\lambda}{2}\binom{r-\lambda}{1}\right] + \binom{\lambda}{1}\binom{b-2r+\lambda}{2} + \binom{r-\lambda}{1}\binom{r-\lambda}{1}\binom{b-2r+\lambda}{1}$

$\Lambda = 2\lambda^2 + 6\lambda - 4b\lambda + 28r^2\lambda - 4r^2 + \lambda b^2 - b\lambda + 2r^2b/2.$

$\Lambda_1 = 2\lambda_1^{\ 2} + 6\lambda_1 - 4b\lambda_1 + 28r^2\lambda_1 - 4r^2 + \lambda b^2 - b\lambda_1 + 2r^2b/2.$

$\Lambda_2 = 2\lambda_2^{\ 2} + 6\lambda_2 - 4b\lambda_2 + 28r^2\lambda_2 - 4r^2 + \lambda b^2 - b\lambda_2 + 2r^2b/2.$

$\Lambda_3 = 2\lambda_3^{\ 2} + 6\lambda_3 - 4b\lambda_3 + 28r^2\lambda_3 - 4r^2 + \lambda b^2 - b\lambda_3 + 2r^2b/2.$

$\Lambda_1(n-1) + \Lambda_2(m-1) + \Lambda_3(mn-n-m+1) = R(K-1) - 2Q_2 - 6Q_3.$

Hence Q.E.D..

Example 3.5.1: Let us consider rectangular BIB design with parameters v = 6, b = 6, r = 3, k = 3, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 1$, m = 2, n = 3 with the association scheme

1 2 3

4 5 6

Applying Theorem 3.5 which is developed as a PBQR design (given in Table 3.5.1).

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1
2	2	2	1	1	1	1	2	2	1	2	2	1	2	1	1	2	1	2	1
2	3	3	2	2	2	2	2	3	1	2	3	2	2	2	2	3	3	2	2
3	3	4	3	2	2	3	3	3	2	2	3	2	3	3	2	3	3	3	3
3	4	4	4	3	3	4	4	4	3	3	3	4	3	3	3	4	4	4	3
4	4	5	5	4	3	4	5	5	4	4	4	4	5	4	4	5	4	4	4
4	5	5	5	5	4	4	5	6	4	5	5	5	5	4	4	5	5	5	5
5	5	6	6	6	5	5	5	6	5	5	6	5	6	6	5	6	6	6	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	6	6	6	6

Table 3.5.1 The number of blocks in PBQR design with parameters V = 6, B = 20, $Q_1 = 9$, $Q_2 = 9$, $Q_3 = 1$, R =30, K = 9, $\Lambda_1 = 42$, $\Lambda_2 = 48$, $\Lambda_3 = 42$, m = 2, n = 3

4 . Applications

The blocks given in Table 3.3.1 can be used for conducting intercropping experiments when the intercrops are sub-divided into various groups based on agronomic practices. We construct design for experiments where each plot consists of two main crops and twelve intercrops such that each of these intercrops is selected from a group of intercrops following Rao and Rao (2001).

Now, let us consider an intercropping experiments using two main crops and twelve intercrops where the intercrops are divided into four groups S_1,S_2,S_3,S_4 with two in each group viz.,S_1 = [1,2], S_2 =[3,4], S_3 = [5,6], S_4 = [7,8], S_5 = [9,10], S_6 =[11,12] .Let us designate the symbols 0, 2 of first row of BTD design with intercrops 1, 2 of S_1 , second row with intercrops 3, 4 of S_2, third row with intercrops 5,6 of S_3 and fourth row of intercrops 7,8 of S_4 fifth row of intercrops 9,10 of S_5 and sixth row of intercrops 11,12 of S_6 .Taking into the consideration the column of the array as the plots of the intercropping experiments in addition to two main crops in each plot, the resulting intercropping experiments will consist of the following six plots on the basis of the blocks given in the Example 3.1.1:

(m_1,m_2,2,4,5,11) ;(m_1,m_2,5,8,10,11) (m_1,m_2,1,4,6,7); (m_1,m_2,2,3,6,9) ; (m_1,m_2,3,8,9,12); (m_1,m_2,1,7,10,12) ;

It is to be noted that the constructed blocks can provide intercropping design with two main crops and 12 intercrops divided into six groups of two intercrops each. Rao and Rao (2001) has developed intercropping design for eight intercrops with one main crop and four intercrops , while this design consists of 12 intercrops merely in six plots having two main crops in each plot. It may be suggested that this design is superior in term of reduction of size of blocks.

In the context of our results, Pandey *et al.* (2003) have studied the effect of maize (*Zea mays* L.) based intercropping system on maize yield as main crop and six intercrops *viz.,* pigeon pea, sesamum, groundnut, blackgram,turmeric and forage *meth* by conducting an experiment during the rainy seasons of 1998 and 1999 at the research farm of Rajendra Agricultural University, Pusa, Samastipur (Bihar). The experiment consisted of six intercrops with one main crops was conducted in randomized complete block design with four replications. Maize was grown at a spacing of 75 cm. Row spacing in sole as well as in intercropping on 26 and 22 June respectively in the first and second year of experimentation. One row of pigeon pea at a distance of 75 cm and 2 rows of other intercrops at 30 cm distance were accommodated between two rows of maize. The intra row spacing of 30, 30, 10, 15, 10, and 15 cm were maintained by thinning for six intercrops.

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