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POSTERIOR ESTIMATES OF EXPONENTIATED MINIMAX DISTRIBUTION UNDER DIFFERENT PRIORS

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Abstract

In this paper, Bayesian estimation of unknown parameter of the exponentiated minimax distribution is examined under different priors. The posterior distributions for the unknown parameter of the exponentiated minimax distribution are derived using the following priors: Jeffrey's, extension of Jeffrey's, gamma-exponential distribution, chi-square-exponential distribution, gamma-exponential-chi-square distribution and chi-square-exponential-inverse Levy distribution. A comparison of these informative and non- informative priors on the basis of posterior variances has also been discussed by making use of simulation techniques.

Key Words: Exponentiated Minimax Distribution, Prior Distribution, Posterior Distribution, R Software.

1. Introduction

The Minimax distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, etc. The exponentiated minimax distribution (EMD) is a generalization to the minimax distribution through adding a new shape parameter $\lambda \in R^{+}$ by exponentiation to distribution function *F*. The three parameters exponentiated minimax distribution (EMD) has the following density function

$$f(x) = \alpha\theta\lambda x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left[1-(1-x^{\alpha})^{\theta}\right]^{\lambda-1}; 0<x<1,\ \theta,\alpha,\lambda>0 \tag{1.1}$$

The corresponding (cdf) of the Exponentiated Minimax distribution is given by

$$F(x)=\left[1-(1-x^{\alpha})^{\theta}\right]^{\lambda}; 0<x<1,\ \theta,\alpha,\lambda>0 \tag{1.2}$$

where α, θ and λ are the shape parameters.

For $\lambda=1$, it represents the minimax distribution, for $\lambda=\alpha=\theta=1$, it represents the Uniform distribution (UD), for $\theta=\lambda=1$, it represents the Power distribution (PD) and for $\alpha=\lambda=1$, it represents the one parameter minimax distribution. Jones (2007) explored the genesis of the minimax distribution and made some similarities between beta and Minimax distributions. Gupta et al. (1998) introduced a class of exponentiated distributions based on cumulative distribution function (CDF) for the exponential distribution. In a similar manner, Nadarajah and Kotz (2006) proposed the exponentiated gamma and exponentiated Gumbel distributions. Double prior selection for discrete case in the case of Poisson distribution is studied by Haq and Aslam (2009) while Sultan and Ahmad (2012) considered the posterior estimates of Poisson distribution using R Software. Radha (2013) studied on the double prior selection for the parameter of Maxwell distribution. Sultan et al. (2014) discussed the Bayesian analysis of power function distribution under double priors. Lanping Li (2015) obtained the Bayes estimates of the shape parameter of minimax distribution under different loss functions especially entropy loss, LINEX loss, precautionary loss, and squared error loss function. Recently, Fatima and Ahmad (2016) considered the characterization and Bayesian estimation of minimax distribution.

2. Maximum Likelihood Estimation for the Shape Parameter λ of Exponentiated Minimax Distribution assuming Shape Parameters α and θ to Be Known

Let us consider a random sample $\underline{x} = (x_1, x_2, ..., x_n)$ of size n from the exponentiated minimax distribution. Then the likelihood function for the given sample observation is

$$L(\underline{x}/\lambda) = \alpha^n \theta^n \lambda^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x^{\alpha})^{\theta-1} \prod_{i=1}^{n} \{1-(1-x^{\alpha})^{\theta}\}^{\lambda-1} \tag{2.1}$$

The log-likelihood function is

$$\begin{aligned} \ln L(\underline{x}/\lambda) = n\ln\alpha + n\ln\theta + n\ln\lambda + (\alpha-1)\sum_{i=1}^{n}\ln x_i + (\theta-1)\sum_{i=1}^{n}\ln(1-x_i^{\alpha}) \\ + (\lambda-1)\sum_{i=1}^{n}\ln\{1-(1-x_i^{\alpha})^{\theta}\} \end{aligned} \tag{2.2}$$

As shape parameter α and θ are assumed to be known, the ML estimator of shape parameter λ is obtained by solving the

$$\frac{\partial}{\partial\lambda} \ln L(\underline{x}/\lambda) == 0 \Rightarrow \hat{\lambda} = -\frac{n}{\sum_{i=1}^{n} \ln\{1-(1-x_i^{\alpha})^{\theta}\}} \tag{2.3}$$

3. Selection of Single Priors for the Unknown Parameter λ of the Exponentiated Minimax Distribution assuming Shape Parameters α and θ to Be Known

3.1 Posterior distribution for the unknown parameter λ of exponentiated Minimax distribution under Jeffrey's Prior Information as single prior

Jeffery's (1964) proposed a formal rule for obtaining a non-informative prior as

$$g_1(\lambda) \propto \sqrt{I(\lambda)}$$

where λ is k-vector valued parameter and $I(\lambda)$ is the Fishers information matrix of order k*k. Thus in our problem the prior distribution of λ to be

$$g_1(\lambda) \propto \frac{1}{\lambda} \quad , \tag{3.1}$$

According to Bayes theorem, the posterior distribution of λ is given by

$$\pi_1(\lambda \mid \underline{x}) \propto L(\underline{x} \mid \lambda)\, g_1(\lambda)$$

Using the prior (3.1) and the likelihood function (2.1), in the above equation we get

$$\pi_1(\lambda \mid \underline{x}) \propto \alpha^n \theta^n \lambda^{n-1} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x^{\alpha})^{\theta-1} \prod_{i=1}^{n} \{1-(1-x^{\alpha})^{\theta}\}^{\lambda-1}$$

$$\pi_1(\lambda \mid \underline{x}) = K \lambda^{n-1} e^{-\lambda T_1} \ , \tag{3.2}$$

where K is independent of λ , $T_1 = \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]^{-1}$ and $K^{-1} = \int_0^{\infty} \lambda^{n-1} e^{-\lambda T_1}\, d\lambda$

$$\Rightarrow \quad K^{-1} = \frac{\Gamma(n)}{T_1^{\,n}} \quad \Rightarrow \quad K = \frac{T_1^{\,n}}{\Gamma(n)}.$$

Using the value of K in (3.2) and the posterior distribution is given by

$$\pi_1(\lambda \mid \underline{x}) = \frac{T_1^{\,n}}{\Gamma(n)} \ \lambda^{n-1} e^{-\lambda T} \quad ; \quad \lambda > 0, \tag{3.3}$$

which is the pdf of gamma distribution with parameters $n_0 = "n"$ and $T_1 = \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]^{-1}$.

3.2 Posterior distribution for EMD under extension of Jeffrey's Prior Information

The extended Jeffrey's prior proposed by Al-Kutubi (2005), is given as:

$g_2(\lambda) \propto [I(\lambda)]^m, m \in R^+, I(\lambda) = -E\left(\frac{d^2}{d\lambda^2} \ln L(\underline{x}/\lambda)\right)$ is the the Fisher Information matrix.

$$\Rightarrow g_2(\lambda) \propto \frac{1}{\lambda^{2m}} \ , \tag{3.4}$$

using the prior (3.4)and the likelihood function (2.1), in the above equation we get

$$\pi_2(\lambda \mid \underline{x}) \propto \alpha^n \theta^n \lambda^{n-2m+1-1} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x^{\alpha})^{\theta-1} \prod_{i=1}^{n} \{1-(1-x^{\alpha})^{\theta}\}^{\lambda-1}$$

$$\pi_2(\lambda \mid \underline{x}) = K \lambda^{n-2m+1-1} e^{-\lambda T_1} \tag{3.5}$$

where K is independent of λ , $T_1 = \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]^{-1}$ and $K^{-1} = \int_0^{\infty} \lambda^{n-2m+1-1} e^{-\lambda T_1}\, d\lambda$

$$\Rightarrow \quad K^{-1} = \frac{\Gamma(n-2m+1)}{T_1^{\,n-2m+1}} \quad \Rightarrow \quad K = \frac{T_1^{\,n-2m+1}}{\Gamma(n-2m+1)}.$$

Using the value of K in (3.5) and the posterior distribution is given by

$$\pi_2(\lambda \mid \underline{x}) = \frac{T_1^{\,n-2m+1}}{\Gamma(n-2m+1)} \ \lambda^{n-2m+1-1} e^{-\lambda T_1} \quad ; \quad \lambda > 0, \tag{3.6}$$

which is the pdf of gamma distribution with parameters $n_1 = (n-2m+1)$ and

$T_1 = \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]^{-1}$.

Thus Baye's estimators for λ under Jeffery's prior and extension of Jeffery's prior are

$$\hat{\lambda}_{JP}=\frac{(n)}{\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]^{-1}}\ ;\hat{\lambda}_{EJP}=\frac{(n-2m+1)}{\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]^{-1}} \tag{3.7}$$

Remark 1.1 Replacing $m=1/2$ in (3.6) we get the same Bayes estimator as obtained in (3.3) corresponding to the Jeffrey's prior , replace $m=3/2$ we get the Hartigan's prior and replace $m=0$ we get the Uniform prior.

4. Comparison of Single Priors with respect to Posterior Variances

The variances of the posterior distribution under all assumed single priors is given by

$$V(\lambda\,|\,X)=\frac{n_i}{T_1^{\,2}};i=0,1. \tag{4.1}$$

5. On the Double Prior Selection for the Unknown Parameter λ of EMD Distribution assuming Shape Parameters α and θ to Be Known

5.1 Posterior distribution of λ using gamma exponential distribution as prior

It is assumed the prior for unknown parameter λ is gamma distribution with hyper parameters a_1 and b_1 as given below.

$$g_3(\lambda)=\frac{b_1^{\,a_1}}{\Gamma a_1}e^{-\lambda b_1}\lambda^{a_1-1}\ ;\quad 0<\lambda<\infty;\ a_1,b_1>0 \tag{5.1}$$

Secondly assume that the prior distribution of λ to be exponential with hyper parameter c_1 .

$$g_4(\lambda)=c_1e^{-\lambda c_1}\ ,\qquad 0<\lambda<\infty;\ c_1>0 \tag{5.2}$$

Now the double prior is defined as

$$g_{11}(\lambda)\propto g_3(\lambda)\ g_4(\lambda)$$
$$g_{11}(\lambda)\propto \lambda^{a_1-1}\,e^{-(b_1+c_1)\lambda} \tag{5.3}$$

The posterior distribution is obtained by combining the equation (5.3) and (2.1) is given as

$$\pi_3(\lambda\,|\,\underline{x})\propto L(\underline{x}\,|\,\lambda)\,g_{11}(\lambda)$$

$$\pi_3(\lambda|\underline{x})\propto \alpha^n\theta^n\lambda^{n+a_1-1}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}(1-x^{\alpha})^{\theta-1}\ \prod_{i=1}^{n}\{1-(1-x^{\alpha})^{\theta}\}^{\lambda-1}\ e^{-(b_1+c_1)\lambda}$$

$$\pi_3(\lambda\,|\,\underline{x})=K\lambda^{n+a_1-1}e^{-\lambda\left(b_1+c_1-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)} \tag{5.4}$$

where K is independent of λ , $T_2=\left(b_1+c_1-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)$ and $K=\frac{T_2^{\,n+a_1}}{\Gamma(n+a_1)}$

Therefore from (5.4) we have

$$\pi_3(\lambda|\underline{x})=\frac{T_2^{\,n+a_1}}{\Gamma(n+a_1)}\lambda^{n+a_1-1}e^{-\lambda T_2}\ ;\lambda>0, \tag{5.5}$$

which is the density function of gamma distribution with parameters $(n_2,\ T_2)$ where $n_2 = (n + a_1)$ and $T_2 = \left(b_1 + c_1 - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right]\right)$.

5.2 Posterior Distribution under Chi-square-Exponential as a prior

Now assuming that the prior distribution of λ is chi-square distribution with hyper parameter d_1 .

$$g_5(\lambda) = \frac{1}{\Gamma(d_1/2)2^{d_1/2}} \lambda^{\frac{d_1}{2}-1} e^{-\frac{\lambda}{2}} \quad ; 0 < \lambda < \infty \ , d_1 > 0 \tag{5.6}$$

Another prior be exponential distribution with hyper parameter c_2 .

$$g_6(\lambda) = \frac{1}{c_2} e^{\frac{-\lambda}{c_2}} \ , \quad 0 < \lambda < \infty;\ c_2 > 0 \tag{5.7}$$

Now the double prior is defined as

$$g_{22}(\lambda) \propto \lambda^{\frac{d_1}{2}-1} e^{-\left(\frac{1}{c_2}+\frac{1}{2}\right)\lambda} \tag{5.8}$$

The posterior distribution is obtained by combining the equation (5.8) and (2.1) is given as

$$\pi_4(\lambda \mid \underline{x}) \propto L(\underline{x} \mid \lambda)\, g_{22}(\lambda)$$

$$\pi_4(\lambda \mid \underline{x}) \propto \alpha^n \theta^n \lambda^{n+\frac{d_1}{2}-1} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1 - x^{\alpha})^{\theta-1} \prod_{i=1}^{n} \{1 - (1 - x^{\alpha})^{\theta}\}^{\lambda-1}\ e^{-\left(\frac{1}{c_2}+\frac{1}{2}\right)\lambda}$$

$$\pi_4(\lambda \mid \underline{x}) = K \lambda^{n+\frac{d_1}{2}-1} e^{-\lambda\left(\frac{1}{c_2}+\frac{1}{2}-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)} \tag{5.9}$$

where K is independent of λ , $T_3 = \left(\frac{1}{c_2} + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right]\right)$ and $K = \frac{T_3^{\,n+\frac{d_1}{2}}}{\Gamma(n + \frac{d_1}{2})}$

Therefore from (5.9) we have

$$\pi_4(\lambda \mid \underline{x}) = \frac{T_3^{\,n+\frac{d_1}{2}}}{\Gamma(n + \frac{d_1}{2})} \lambda^{n+\frac{d_1}{2}-1} e^{-\lambda T_3}, \tag{5.10}$$

which is the density function of gamma distribution with parameters $(n_3,\ T_3)$ where $n_3 = (n + \frac{d_1}{2})$ and $T_3 = \left(\frac{1}{c_2} + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right]\right)$.

Thus, Bayes estimators for λ under gamma-exponential Prior and chi-square-exponential prior are

$$\hat{\lambda}_{GEP} = \frac{(n + a_1)}{\left(b_1 + c_1 - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right]\right)} \quad ; \hat{\lambda}_{CEP} = \frac{\left(n + \frac{d_1}{2}\right)}{\left(\frac{1}{c_2} + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right]\right)}$$

6. Comparison of double priors with respect to Posterior Variances

The variances of the posterior distribution under all of assumed informative priors are calculated by assuming different set of values for hyper parameters, different sample size and different value of parameter which is given by

$$V(\lambda / X) = \frac{n_i}{T_i^2}; i = 2,3. \qquad (6.1)$$

7. Selection of Triple Priors for the Unknown Parameter λ of the Exponentiated Minimax Distribution assuming Shape Parameters α and θ to Be Known

7.1 Posterior distribution of unknown parameter λ using gamma- exponential -chi square prior

We assume that the prior distribution of λ is gamma distribution with hyper parameters a_2 and b_2 as given below:

$$g_7(\lambda) = \frac{b_2^{a_2}}{\Gamma a_2} e^{-\lambda b_2} \lambda^{a_2 - 1} \ ; \ 0 < \lambda < \infty; \ a_2, b_2 > 0 \qquad (7.1)$$

The second prior assumed is exponential distribution with hyper parameter c_3 and is given by:

$$g_8(\lambda) = c_3 e^{-\lambda c_3} \ , \ 0 < \lambda < \infty; \ c_3 > 0 \qquad (7.2)$$

The third prior assumed is chi-square distribution with hyper parameter d_2 and is given by:

$$g_9(\lambda) = \frac{1}{\Gamma(d_2/2) 2^{d_2/2}} \lambda^{\frac{d_2}{2} - 1} e^{-\frac{\lambda}{2}} \ ; \ 0 < \lambda < \infty \ , d_2 > 0 \qquad (7.3)$$

Now we define a triple prior for λ by combining the above three priors as given in (7.1), (7.2) and (7.3) as follows:

$$g_{111}(\lambda) \propto g_7(\lambda) \ g_8(\lambda) \ g_9(\lambda)$$

$$g_{111}(\lambda) \propto \lambda^{a_2 + \frac{d_2}{2} - 1} e^{-(b_2 + c_3 + \frac{1}{2})\lambda} \qquad (7.4)$$

The posterior distribution of unknown parameter λ using gamma –exponential-chi square prior is obtained by combining (7.4) and (2.1) given as:

$$\pi_5(\lambda | \underline{x}) \propto \alpha^n \theta^n \lambda^{n + a_2 + \frac{d_2}{2} - 1} \prod_{i=1}^{n} x_i^{\alpha - 1} \prod_{i=1}^{n} (1 - x^{\alpha})^{\theta - 1} \prod_{i=1}^{n} \{1 - (1 - x^{\alpha})^{\theta}\}^{\lambda - 1} e^{-\left(b_2 + c_3 + \frac{1}{2}\right)\lambda}$$

$$\pi_5(\lambda | \underline{x}) = K \lambda^{n + a_2 + \frac{d_2}{2} - 1} e^{-\lambda \left(b_2 + c_3 + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right] \right)} \qquad (7.5)$$

where K is independent of λ , $T_4 = \left(b_2 + c_3 + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1 - (1 - x_i^{\alpha})^{\theta}\right] \right)$ and

$$K = \frac{T_4^{n + a_2 + \frac{d_2}{2}}}{\Gamma(n + a_2 + \frac{d_2}{2})}$$

Therefore from (7.5) we have

$$\pi_5(\lambda \mid \underline{x}) = \frac{T_4^{\,n+a_2+\frac{d_2}{2}}}{\Gamma(n+a_2+\frac{d_2}{2})}\lambda^{n+a_2+\frac{d_2}{2}-1}e^{-\lambda T_4};\quad \lambda>0, \tag{7.6}$$

which is the density function of gamma distribution with parameters $(n_4,\ T_4)$ where $n_4=(n+a_2+\frac{d_2}{2})$ and $T_4=\left(b_2+c_3+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)$.

7.2 Posterior distribution of unknown parameter λ using chi square- exponential-inverse Levy distributions

We assume that the prior distribution of λ is chi square distribution with hyper parameter d_3

$$g_{10}(\lambda)=\frac{1}{\Gamma(d_3/2)2^{d_3/2}}\lambda^{\frac{d_3}{2}-1}e^{-\frac{\lambda}{2}}\ ;\qquad 0<\lambda<\infty\ ,d_3>0 \tag{7.7}$$

The second prior assumed is exponential distribution with hyper parameter c_4 and is given by:

$$g_{11}(\lambda)=c_4e^{-\lambda c_4}\ ,\qquad 0<\lambda<\infty;\ c_4>0 \tag{7.8}$$

The third prior assumed is inverse levy distribution with hyper parameter d_4 and is given by:

$$g_{12}(\lambda)=\sqrt{\frac{d_4}{2\pi}}\lambda^{-\frac{1}{2}}e^{\frac{-d_4\lambda}{2}}\ ;\quad 0<\lambda<\infty\ ,d_4>0 \tag{7.9}$$

Now we define a triple prior for λ by combining the above three priors as given in (7.7), (7.8) and (7.9) as follows:

$$g_{222}(\lambda)\propto\lambda^{\frac{d_3-1}{2}-1}e^{-(c_4+\frac{d_4}{2}+\frac{1}{2})\lambda} \tag{7.10}$$

The posterior distribution of unknown parameter λ using gamma –exponential-chi square prior is obtained by combining (7.10) and (2.1) given as:

$$\pi_5(\lambda\mid\underline{x})\propto\alpha^n\theta^n\lambda^{n+\frac{d_3-1}{2}-1}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}(1-x^{\alpha})^{\theta-1}\prod_{i=1}^{n}\{1-(1-x^{\alpha})^{\theta}\}^{\lambda-1}e^{-\left(c_4+\frac{d_4}{2}+\frac{1}{2}\right)\lambda}$$

$$\pi_5(\lambda\mid\underline{x})=K\lambda^{n+\frac{d_3-1}{2}-1}e^{-\lambda\left(c_4+\frac{d_4}{2}+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)} \tag{7.11}$$

where K is independent of λ, $T_5=\left(c_4+\frac{d_4}{2}+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)$ and

$$K=\frac{T_5^{\left(n+\frac{d_3-1}{2}\right)}}{\Gamma\left(n+\frac{d_3-1}{2}\right)}$$

Therefore from (7.11) we have

$$\pi_5(\lambda|\underline{x}) = \frac{T_5^{\left(n+\frac{d_3-1}{2}\right)}}{\Gamma\left(n+\frac{d_3-1}{2}\right)} \lambda^{\left(n+\frac{d_3-1}{2}\right)-1} e^{-\lambda T_5}; \quad \lambda > 0, \tag{7.12}$$

which is the density function of gamma distribution with parameters $(n_5,\ T_5)$ where $n_5 = \left(n + \frac{d_3-1}{2}\right)$ and $T_5 = \left(c_4 + \frac{d_4}{2} + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)$.

Thus, Bayes estimators for λ under gamma –exponential-chi square prior and exponential-chi- square- inverse levy distribution are

$$\hat{\lambda}_{GECP} = \frac{(n + a_2 + \frac{d_2}{2})}{\left(b_2 + c_3 + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)} \quad ; \hat{\lambda}_{ECILP} = \frac{\left(n + \frac{d_3-1}{2}\right)}{\left(c_4 + \frac{d_4}{2} + \frac{1}{2} - \sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)}$$

8. Comparison of Triple priors with respect to Posterior Variances

The variances of the posterior distribution under all of assumed informative priors are calculated by assuming different set of values for hyper parameters, different sample size and different value of parameter which is given by

$$V(\lambda / X) = \frac{n_i}{T_i^2}; i = 4{,}5. \tag{8.1}$$

The posterior mean and posterior variance of parameter λ are presented in Table 1 for EMD under different types of priors.

Type of Prior	Prior Distribution	Posterior Mean	Posterior Variance
Jeffrey's Prior	$\frac{1}{\lambda}$	$\frac{(n)}{-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]}$	$\frac{(n)}{\left\{-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right\}^2}$
Extension of Jeffrey's Prior	$\frac{1}{\lambda^{2m}}$	$\frac{(n-2m+1)}{-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]}$	$\frac{(n-2m+1)}{\left\{-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right\}^2}$
Gamma-Exponential Distribution	$\lambda^{a_1-1} e^{-(b_1+c_1)\lambda}$	$\frac{(n+a_1)}{\left(b_1+c_1-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)}$	$\frac{(n+a_1)}{\left(b_1+c_1-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)^2}$
Chi-Square-Exponential Distribution	$\lambda^{\frac{d_1}{2}-1} e^{-\left(\frac{1}{c_2}+\frac{1}{2}\right)\lambda}$	$\frac{\left(n+\frac{d_1}{2}\right)}{\left(\frac{1}{c_2}+\frac{1}{2}-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)}$	$\frac{\left(n+\frac{d_1}{2}\right)}{\left(\frac{1}{c_2}+\frac{1}{2}-\sum_{i=1}^{n} \ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)^2}$

Gamma-Exponential-Chi-Square Distribution	$\lambda^{a_2+\frac{d_2}{2}-1} e^{-(b_2+c_3+\frac{1}{2})\lambda}$	$\dfrac{(n+a_2+\frac{d_2}{2})}{\left(b_2+c_3+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)}$	$\dfrac{(n+a_2+\frac{d_2}{2})}{\left(b_2+c_3+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)^2}$
Chi-Square Exponential Inverse Levy Distribution	$\lambda^{\frac{d_3-1}{2}-1} e^{-(c_4+\frac{d_4}{2}+\frac{1}{2})\lambda}$	$\dfrac{\left(n+\frac{d_3-1}{2}\right)}{\left(c_4+\frac{d_4}{2}+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)}$	$\dfrac{\left(n+\frac{d_3-1}{2}\right)}{\left(c_4+\frac{d_4}{2}+\frac{1}{2}-\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]\right)^2}$

Table 1: Posterior mean and posterior variance of unknown parameter λ for EMD with different priors

9 Simulation Study and Data Analysis

9.1 Simulation study

In our simulation study, we choose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The parameter λ is estimated for exponentiated minimax distribution by using Bayesian method of estimation under various types of priors. For the parameter λ we have considered $\lambda = 2.0, 2.5$ and 3.0. The parameters α and θ has been fixed at 0.5 and 1.0. The value of Jeffrey's extension was $m = 2$ and the values of hyper parameters were a_i, b_i, c_i and d_i =1.0, 1.5, and 2.0. This was iterated 10000 times and the parameter λ for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes by using various types of priors. The results are presented in tables given below:

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Jeffrey's prior	Extension Jeffrey's prior m=2	Gamma Exponential Distribution
0.5	1.0	2.0	1.0	Mean	1.82670	1.60750	1.65754
				post.var	0.13347	0.11745	0.10567
			1.5	Mean	1.39619	1.22864	1.26758
				post.var	0.10908	0.09599	0.08054
			2.0	Mean	2.12336	1.86856	1.71170
				post.var	0.18034	0.15870	0.10851
0.5	1.0	2.5	1.0	Mean	3.10621	2.73347	2.58748
				post.var	0.38594	0.33962	0.25750
			1.5	Mean	2.66361	2.34398	2.13955
				post.var	0.28379	0.24973	0.17274
			2.0	Mean	2.60560	2.29293	1.98606
				post.var	0.27156	0.23897	0.14609

0.5	1.0	3.0	1.0	Mean	4.06607	3.57814	3.19079
				post.var	0.66131	0.58196	0.39158
			1.5	Mean	2.66372	2.34407	2.13962
				post.var	0.28381	0.24975	0.17275
			2.0	Mean	2.56247	2.25497	1.96275
				post.var	0.26265	0.23113	0.14268

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Chi-Square Exponential Distribution	Gamma-Exponential-Chi-Square Distribution	Chi-Square-Exponential Inverse levy Distribution
0.5	1.0	2.0	1.0	Mean	1.67919	1.63723	1.59379
				post.var	0.11057	**0.10115**	0.10160
			1.5	Mean	1.35011	1.27301	1.22241
				post.var	0.09685	**0.07843**	0.07890
			2.0	Mean	2.03542	1.72056	1.66952
				post.var	0.15934	**0.10572**	0.10930
0.5	1.0	2.5	1.0	Mean	2.67061	2.51223	2.48796
				post.var	0.27969	**0.23816**	0.24759
			1.5	Mean	2.44020	2.11474	2.08063
				post.var	0.23124	**0.16411**	0.17144
			2.0	Mean	2.45405	1.98656	1.94735
				post.var	0.23163	**0.14094**	0.14871
0.5	1.0	3.0	1.0	Mean	3.33401	3.06413	3.06807
				post.var	0.43590	**0.35429**	0.37652
			1.5	Mean	2.44029	2.11480	2.08069
				post.var	0.23126	**0.16412**	0.17145
			2.0	Mean	2.41721	1.96405	1.92362
				post.var	0.22472	**0.13776**	0.14511

Table 2: Posterior Mean and Posterior Variance for EMD using different priors with n=25

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Jeffrey's prior	Extension Jeffrey's prior m=2	Gamma Exponential Distribution
0.5	1.0	2.0	1.0	Mean	1.68465	1.58357	1.60986
				post.var	0.07654	0.07195	0.06715
			1.5	Mean	2.29435	2.15669	2.07722
				post.var	0.09670	0.09090	0.07774
			2.0	Mean	2.35670	2.21530	2.06218
				post.var	0.06225	0.05852	0.04972
0.5	1.0	2.5	1.0	Mean	2.17676	2.04619	2.04246
				post.var	0.05521	0.05190	0.04952
			1.5	Mean	2.49778	2.34791	2.23740
				post.var	0.16901	0.15886	0.12621
			2.0	Mean	2.44443	2.29776	2.12638
				post.var	0.09931	0.09335	0.07439
0.5	1.0	3.0	1.0	Mean	3.66448	3.44461	3.25993
				post.var	0.17953	0.16876	0.14602
			1.5	Mean	3.59132	3.37584	3.04329
				post.var	0.17316	0.16277	0.12884
			2.0	Mean	2.97338	2.79497	2.49809
				post.var	0.17636	0.16578	0.11976

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Chi-Square Exponential Distribution	Gamma-Exponential-Chi-Square Distribution	Chi-Square-Exponential Inverse levy Distribution
0.5	1.0	2.0	1.0	Mean	1.61964	1.60039	1.57830
				post.var	0.06897	0.06541	0.06583
			1.5	Mean	2.21043	2.06581	2.04745
				post.var	0.08880	0.07589	0.07734
			2.0	Mean	2.29564	2.06097	2.04320
				post.var	0.05924	0.04914	0.04981
0.5	1.0	2.5	1.0	Mean	2.06376	2.02199	2.00241
				post.var	0.05059	0.04848	0.04854
			1.5	Mean	2.39563	2.22172	2.20707
				post.var	0.15044	0.12193	0.12625
			2.0	Mean	2.37710	2.12385	2.10815
				post.var	0.09283	0.07303	0.07506
0.5	1.0	3.0	1.0	Mean	3.33454	3.18994	3.19601
				post.var	0.15265	0.13987	0.14316
			1.5	Mean	3.36335	2.99900	3.01395
				post.var	0.15390	0.12442	0.12892
			2.0	Mean	2.86261	2.48640	2.48574
				post.var	0.16028	0.11641	0.12209

Table 3: Posterior Mean and Posterior Variance for EMD using different priors with n=50

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Jeffrey's prior	Extension Jeffrey's prior m=2	Gamma Exponential Distribution
0.5	1.0	2.0	1.0	Mean	1.82645	1.77166	1.77970
				post.var	0.02826	0.02742	0.02672
			1.5	Mean	1.69484	1.64400	1.63703
				post.var	0.03625	0.03516	0.03293
			2.0	Mean	1.94288	1.88460	1.83883
				post.var	0.03093	0.03001	0.02754
0.5	1.0	2.5	1.0	Mean	2.42860	2.35575	2.33927
				post.var	0.06933	0.06725	0.06320
			1.5	Mean	2.64045	2.56123	2.48334
				post.var	0.07773	0.07540	0.06719
			2.0	Mean	2.35687	2.28616	2.19689
				post.var	0.06561	0.06364	0.05505
0.5	1.0	3.0	1.0	Mean	2.99243	2.90266	2.85168
				post.var	0.08834	0.08569	0.07949
			1.5	Mean	2.99131	2.90157	2.78615
				post.var	0.07511	0.07286	0.06509
			2.0	Mean	2.56837	2.49132	2.37567
				post.var	0.09975	0.09675	0.08020

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Chi-Square Exponential Distribution	Gamma-Exponential-Chi-Square Distribution	Chi-Square-Exponential Inverse levy Distribution
0.5	1.0	2.0	1.0	Mean	1.78663	1.77289	1.76208
				post.var	0.02703	**0.02642**	0.02646
			1.5	Mean	1.67445	1.63593	1.62342
				post.var	0.03495	**0.03258**	0.03281
			2.0	Mean	1.92491	1.84027	1.82827
				post.var	0.03017	**0.02736**	0.02758
0.5	1.0	2.5	1.0	Mean	2.35496	2.32393	2.31611
				post.var	0.06449	**0.06195**	0.06257
			1.5	Mean	2.58075	2.47146	2.46785
				post.var	0.07346	**0.06597**	0.06722
			2.0	Mean	2.32562	2.19479	2.18815
				post.var	0.06299	**0.05433**	0.05553
0.5	1.0	3.0	1.0	Mean	2.87820	2.82591	2.82345
				post.var	0.08136	**0.07769**	0.07871
			1.5	Mean	2.91212	2.76874	2.77086
				post.var	0.07106	**0.06394**	0.06512
			2.0	Mean	2.52909	2.37135	2.36831
				post.var	0.09467	**0.07876**	0.08128

Table 4: Posterior Mean and Posterior Variance for EMD using different priors with n=100

9.2 A real data example

In this section, we analyze the real life data set is given by Dumonceaux and Antle (1973) and it represents the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania, over 20 four-year periods (1890 to 1969) as 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Jeffrey's prior	Extension Jeffrey's prior m=2	Gamma Exponential Distribution
0.5	1.0	2.0	1.0	Mean	2.22316	1.88969	1.90975
				post.var	0.24712	0.21005	0.17367
			1.5	Mean	2.22316	1.88969	1.79223
				post.var	0.24712	0.21005	0.14940
			2.0	Mean	2.22316	1.88969	1.69281
				post.var	0.24712	0.21005	0.13025
0.5	1.0	2.5	1.0	Mean	2.22316	1.88969	1.90975
				post.var	0.24712	0.21005	0.17367
			1.5	Mean	2.22316	1.88969	1.79223
				post.var	0.24712	0.21005	0.14940
			2.0	Mean	2.22316	1.88969	1.69281
				post.var	0.24712	0.21005	0.13025
0.5	1.0	3.0	1.0	Mean	2.22316	1.88969	1.90975
				post.var	0.24712	0.21005	0.17367
			1.5	Mean	2.22316	1.88969	1.79223
				post.var	0.24712	0.21005	0.14940
			2.0	Mean	2.22316	1.88969	1.69281
				post.var	0.24712	0.21005	0.13025

α	θ	λ	Hyper Parameters $a_i=b_i=c_i=d_i$	Mean/P.V	Chi-Square Exponential Distribution	Gamma-Exponential-Chi-Square Distribution	Chi-Square-Exponential Inverse levy Distribution
0.5	1.0	2.0	1.0	Mean	1.95309	1.87018	1.81881
				post.var	0.18607	**0.16267**	0.16540
			1.5	Mean	2.04175	1.78054	1.72396
				post.var	0.20090	**0.14248**	0.14676
			2.0	Mean	2.10080	1.70418	1.64050
				post.var	0.21016	**0.12627**	0.13128
0.5	1.0	2.5	1.0	Mean	1.95309	1.87018	1.81881
				post.var	0.18607	**0.16267**	0.16540
			1.5	Mean	2.04175	1.78054	1.72396
				post.var	0.20090	**0.14248**	0.14676
			2.0	Mean	2.10080	1.70418	1.64050
				post.var	0.21016	**0.12627**	0.13128

0.5	1.0	3.0	1.0	Mean	1.95309	1.87018	1.81881
				post.var	0.18607	**0.16267**	0.16540
			1.5	Mean	2.04175	1.78054	1.72396
				post.var	0.20090	**0.14248**	0.14676
			2.0	Mean	2.10080	1.70418	1.64050
				post.var	0.21016	**0.12627**	0.13128

Table 5: Posterior Mean and Posterior Variance for EMD using different priors with real data example

The posterior mean and posterior variance under all the assumed priors are calculated by assuming the different values of hyper parameters. From tables 2 to 5, it is clear that the posterior variance under the triple prior Gamma-Exponential-Chi-Square distribution is less as compared to other assumed priors, which shows that this prior is efficient as compared to other priors and this less variation in posterior distribution helps in making more precise Bayesian estimation of the true unknown parameter λ of exponentiated minimax distribution.

10. Conclusion and Discussion

In this paper, we have used two single priors, two double priors and two triple priors for comparison. In single prior, we have only single information to analyze, in double prior we have more information than single prior, while as in triple prior we have more information than single prior and double prior. We compared all three types of priors and at the end of study we find the triple prior is more efficient than double prior and single prior and during the comparison we got better Bayes estimates and posterior variances by using triple prior gamma-exponential-chi-square distribution

The Posterior estimates of exponentiated minimax distribution under different priors are considered using simulation study. From the results, we observe that, Bayes Estimator under the triple prior gamma-exponential-chi-square distribution has the less posterior variance. So we can say that the triple prior gamma-exponential-chi-square provides the minimum posterior variance as compared to the other assumed priors which are compared in this manuscript.

It is also observed that the real life data also confirms the simulated data results. Therefore, we conclude that the gamma-exponential-chi-square distribution performs well in the exponentiated minimax distribution.

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