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BAYESIAN ANALYSIS OF DAGUM DISTRIBUTION

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Abstract

The Dagum distribution is very useful to represent the distribution of income, actuarial, meteorological data as well for survival analysis. Moreover, it is considered to be the most suitable choice as compared to other three parameter distributions in several cases. It belongs to the generalized beta distribution and is generated from generalized beta-II by considering a shape parameter one and referred as inverse Burr distribution. In this paper, we obtain Bayesian estimation of the scale parameter of the Dagum distribution under informative and non-informative prior. Bayes estimators are derived using different loss functions. These estimators are compared using risk functions.

Key Words: Dagum Distribution, Mixture Distribution, Predictive Distribution, Extension of Jeffreys Prior, Loss Functions, Risk Function, Efficiency.

1. Introduction

The Dagum distribution was proposed by Camilo Dagum in a series of papers in the 1970s to fit heavy tailed models present in empirical income and wealth distributions as well as permitting an interior mode. The classical distributions, such as Pareto distribution and the lognormal distribution, were used to summarize such data. The former aspect is well captured by the Pareto but not by the log-normal distribution, the latter by the log-normal but not the Pareto distribution. In this paper, the problem of Bayesian analysis of Dagum distribution for the complete sample is studied. The maximum likelihood and Bayes estimators of one shape parameter p under different priors using different loss functions are obtained by assuming other shape and scale parameters to be known. Also, these estimators are compared using risk functions in the simulation study. The estimates having minimum risk are considered to be better.

The CDF of Dagum distribution is given by

$$F(x)=\left[1+\left(\frac{x}{b}\right)^{-a}\right]^{-p} \qquad ,x>0. \tag{1.1}$$

The parameters a and p are shape parameters and b is a scale parameter.

The Dagum distribution arose from several variants of a new model on the size distribution of personal income and is mostly associated with the study of income distribution. There are two forms of Dagum distribution namely a three-parameter specification (Type I) and a four-parameter specification (Type II). A summary of the genesis of this distribution can be found in "A Guide to the Dagum Distributions". Dagum (1983) refers to his system as the generalized logistic-Burr system. This is due to the fact that the Dagum distribution with p=1 is also known as the log-logistic distribution (the model Dagum 1975 experimented with). The Dagum distribution is a Burr III distribution with an additional scale parameter and therefore a rediscovery of a distribution that had been known for some 30 years prior to its introduction in economics. However, it is not the only rediscovery of this distribution. Mielke (1973), in a meteorological application, arrives at a three-parameter distribution he calls the kappa distribution. It amounts to the Dagum distribution in a different parametrization. Mielke and Johnson (1974) refer to it as the beta-K distribution. Even in the income distribution literature there is a parallel development: Fattorini and Lemmi (1979), starting from Mielke's kappa distribution but apparently unaware of Dagum (1977), propose (1.1) as an income distribution and fit it to several data sets, mostly from Italy.

An interesting aspect of Dagum distribution is that it admits a mixture representation in terms of generalized gamma and inverse Weibull distributions. The Dagum distribution can be obtained as a compound generalized gamma (*GG*) distribution whose scale parameter follows an inverse Weibull (*IW*) distribution with identical shape parameters.

The *GG* and *IW* distributions have the PDFs given by

$$f_{GG}(x)=\frac{a}{\theta^{a\,p}\,\Gamma(p)}x^{a\,p-1}e^{-\left(\frac{x}{\theta}\right)^{a}} \qquad ,x>0,(a,p,\theta)>0, \tag{1.2}$$

and $$f_{IW}=a\left(\frac{b}{\theta}\right)^{a+1}e^{\left(\frac{b}{\theta}\right)^{a+1}} \quad ;\theta>0,a,b>0. \tag{1.3}$$

Thus, the mixture distribution that results from marginalizing over θ is given by

$$\begin{aligned} f_D(x) &= \int_0^\infty f_{GG}(x\mid\theta)f_{IW}(\theta)d\theta \\ &= \frac{a^2\,b^a}{\Gamma(p)}x^{a\,p-1}\int_0^\infty \frac{1}{\theta^{a(p+1)+1}}e^{-\left(b^a+x^a\right)\frac{1}{\theta^a}}\,d\theta \\ &= \frac{a\,p}{b^{a\,p}}x^{a\,p-1}\left[1+\left(\frac{x}{b}\right)^a\right]^{-(p+1)}. \end{aligned} \tag{1.4}$$

The r^{th} moment of X is given by

$$E(X)^r=\frac{\Gamma\left(p+\frac{r}{a}\right)\Gamma\left(1-\frac{r}{a}\right)}{\Gamma(p)}. \tag{1.5}$$

Domma, Giordano and Zenga (2011) and Domma (2007) estimated the parameters of Dagum distribution with censored samples and by the right-truncated Dagum distribution respectively by maximum likelihood estimation. McGarvey, et al. (2002)

studied the estimation and skewness test for the Dagum distribution. Shahzad and Asghar (2013) estimated the parameter of this distribution by TL-moments. Oluyede and Rajasooriya (2013) introduced the Mc-Dagum distribution.

2. Maximum Likelihood Estimation

We assume that $X=(x_1,x_2,...,x_n)$ is a random sample from $X\sim D(a,b,p)$, then the likelihood function of the scale parameter, *p* (keeping *a* and *b* fixed) is given by

$$L=\left(\frac{a\,p}{b^{a\,p}}\right)^n\prod_{i=1}^{n}x_i^{\,a\,p-1}\prod_{i=1}^{n}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}. \tag{2.1}$$

The log-likelihood equation is given by

$$l=\ln L=n\left(\ln a+\ln p-a\,p\ln b\right)+\left(a\,p-1\right)\sum_{i=1}^{n}\ln x_i-\left(p+1\right)\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^a\right] \tag{2.2}$$

Thus the MLE is obtained by the equation

$$\frac{\partial l}{\partial p}=\frac{n}{p}-n\,a\ln b+a\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^a\right]=0$$

$$\Rightarrow\qquad \frac{n}{p}=\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^a\right]-\sum_{i=1}^{n}\ln\left(\frac{x_i}{b}\right)^a$$

$$\therefore\qquad \hat{p}=\frac{n}{\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^{-a}\right]}=\frac{n}{T},\qquad T=\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^{-a}\right] \tag{2.3}$$

3. Posterior Distribution of $p|x$ under Different Prior Distributions

In this section, we present posterior distribution of *p* under an informative prior, i.e., Mukherjee-Islam (MI) prior and a non-informative, i.e., extension of Jeffreys prior. The likelihood function (2.1) of *p* can be written in terms of *T* as

$$L(a,b,p)\propto p^n\prod_{i=1}^{n}\left(\frac{x_i}{b^{ap}}\right)\prod_{i=1}^{n}\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}$$

$$=p^n\,e^{p\sum_{i=1}^{n}\ln\left(\frac{x_i}{b}\right)^a}\,e^{-p\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^a\right]}$$

$$\Rightarrow\qquad L(p\,|\,x)\propto p^n e^{-p\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^{-a}\right]}=p^n e^{-pT}$$

where $T=\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{b}\right)^{-a}\right]$ (3.1)

3.1 Posterior distribution under Mukherjee-Islam prior

Assume that p has a Mukherjee-Islam prior with hyper parameters (α_1, σ) > 0 defined by

$$g_1(p) = \alpha_1 \sigma^{-\alpha_1} p^{\alpha_1 - 1} \quad ; p > 0, \alpha_1 > 0, \sigma > 0. \tag{3.2}$$

Then the posterior distribution of p is given by

$$\pi_2(p|x) \propto p^{n+\alpha_1-1} e^{-pT}$$

$$= K p^{n+\alpha_1-1} e^{-pT}$$

where K is the normalized constant given by

$$K = \int_0^\infty p^{n+\alpha_1-1} e^{-pT} dp = \frac{\Gamma(n+\alpha_1)}{T^{n+\alpha_1}}.$$

Thus, the posterior distribution of $p|x$ is given by

$$\pi_1(p \mid x) = \frac{T^{n+\alpha_2}}{\Gamma(n+\alpha_1)} p^{n+\alpha_1-1} e^{-pT} \tag{3.3}$$

which is a gamma density with parameters $T, \beta_1 = n + \alpha_1$.

3.2 Posterior distribution under extension of Jeffreys prior

The extension of Jeffreys prior is defined by

$$g_2(p) \propto [I(p)]^m ; m > 0.$$

where $[I(p)]$ is the Fisher Information given by:

$$[I(p)] = -E\left[\frac{\partial^2 l}{\partial p^2}\right] = \frac{n}{p^2},$$

where l is the log-likelihood defined in (2.2). Thus, the extension of Jeffreys prior is given by

$$g_2(p) \propto \frac{1}{p^{2m}}, m > 0. \tag{3.4}$$

The posterior distribution is defined by

$$\pi_2(p|x) \propto p^{n-2m} e^{-pT} = K p^{n-2m} e^{-pT},$$

where K is the normalized constant given by

$$K = \int_0^\infty p^{n-2m} e^{-pT} dp = \frac{\Gamma(n-2m+1)}{T^{n-2m+1}}.$$

Thus the posterior distribution of $p|x$ is given by

$$\pi_2(p \mid x) = \frac{T^{n-2m+1}}{\Gamma(n-2m+1)} p^{n-2m} e^{-pT}, \tag{3.5}$$

which is a gamma density with parameters T, $\beta_2 = n - 2m + 1$.

4. Bayes Estimators of p|x under Different Priors using Different Loss Functions

The choice of the loss function is very important in decision analysis. In some estimation problems overestimation may be more serious than underestimation, or vice-versa, see Parsian and Kirmani (2002) and the references there in. In such cases, the usual methods of estimation may be inappropriate. To deal with such cases, a useful and flexible class of asymmetric loss functions were introduced. In this section we introduce the K-loss function, entropy loss function and squared logarithmic loss function.

We have already discussed in section (3) that using Mukherjee Islam prior and extension of Jeffreys prior given by (3.3) and (3.5) respectively leads to posterior distribution given by the Gamma density:

$$\pi_i(p\,|\,x)=\frac{T^{\beta_i}}{\Gamma(\beta_i)}p^{\beta_i-1}e^{-pT}, \qquad i=1,2,$$

where θ_i and β_i $(i=1,2)$ are shape and rate parameters of gamma distribution defined in (3.3) and (3.5) respectively. Below are presented the loss functions- K-loss function, entropy loss function and squared logarithmic loss function and the Bayes estimators using these loss functions.

4.1 K-loss function (KLF)

Wasan (1970) proposed the K-loss function (KLF) that is used as a measure of inaccuracy for an estimator of a scale parameter of a distribution defined on R^+ (0, ∞) by

$$l(\hat{p},p)=\frac{(\hat{p}-p)^2}{\hat{p}\,p} \tag{4.1}$$

Using K-loss function (K-LF), we have

$$E(p\,|\,x)=\frac{\beta_i}{T}, \;\; \& \;\; E\left(p^{-1}\,|\,x\right)=\frac{T}{\beta_i-1}.$$

Thus, $$\hat{p}_{KL}=\sqrt{\frac{E(p\,|\,x)}{E\left(p^{-1}\,|\,x\right)}}=\frac{\sqrt{\beta_i\,(\beta_i-1)}}{T}. \tag{4.2}$$

4.2 Entropy loss function (ELF)

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{p}}{p}$. In this case, the loss function used by Dey et al. (1987) is given by

$$l(\hat{p},p)=\left(\frac{\hat{p}}{p}\right)-\log\left(\frac{\hat{p}}{p}\right)-1 \tag{4.3}$$

whose minimum occurs at $\hat{p}=p$. The Bayes estimator under the entropy loss function (ELF) is denoted by $\hat{p}_{EL}$ and defined by

$$\hat{p}_{EL} = \left[E\left(p^{-1} \mid x\right)\right]^{-1} = \frac{\beta_i - 1}{T}. \tag{4.4}$$

4.3 Squared logarithmic loss function (SLLF)

The squared-log error loss function has the form

$$l\left(\hat{p}, p\right) = \left[\log\left(\frac{\hat{p}}{p}\right)\right]^2 = \left(\log \hat{p} - \log p\right)^2 \tag{4.5}$$

The loss function is convex if $\hat{p}/p \le e$ and concave otherwise, and its risk function has a unique minimum w.r.t. *p*. The Bayes estimator using SLLF is denoted by $\hat{p}_{SL}$ and is given by

$$\hat{p}_{SL} = \exp\left[E\left(\ln p \mid x\right)\right]$$

In this case, Using squared logarithmic loss function (SLLF), we have

$$\ln \hat{p} = E\left(\ln p \mid x\right) = \psi\left(\beta_i\right) - \ln \theta_i .$$

$$\therefore \quad \hat{p}_{SL} = \frac{e^{\psi(\beta_i)}}{\theta_i}. \tag{4.6}$$

Table 1 shows Bayes estimators under MI prior and extended Jeffreys prior using K-loss, entropy loss and squared logarithmic loss function:

	Mukherjee-Islam Prior	Extension of Jeffreys Prior
	$T, \beta_1 = n + \alpha_1$	$T, \beta_2 = n - 2m + 1$
$\hat{p}_{KL}$	$\frac{\sqrt{\left(n + \alpha_2\right)\left(n + \alpha_2 - 1\right)}}{T}$	$\frac{\sqrt{\left(n - 2m + 1\right)\left(n - 2m\right)}}{T}$
$\hat{p}_{EL}$	$\frac{n + \alpha_2 - 1}{T}$	$\frac{n - 2m + 1}{T}$
$\hat{p}_{SL}$	$\frac{e^{\psi\left(n + \alpha_2\right)}}{T}$	$\frac{e^{\psi\left(n - 2m + 1\right)}}{T}$

Table 1: Bayes Estimators of *p* under MI and extended Jeffreys Prior

5. Posterior Variances under Different Priors Distributions

The variances of the posterior distribution under the informative and non-informative prior are calculated by assuming different set of values for hyper parameters, different sample size and different value of parameter which is given by

$$V\left(\alpha \mid x\right) = \frac{\beta_i}{T^2}, i = 1,2, \tag{5.1}$$

where T and β_i $(i = 1,2)$ are shape and rate parameters of gamma distribution defined in (3.3) and (3.5) respectively.

6. Efficiency of the Estimators

Since X follows a Dagum distribution with parameters (b, a, p), then $T = \sum_{i=1}^{n} \ln\left[1 + \left(\frac{x_i}{b}\right)^{-a}\right]$ is distributed as a Gamma variate with parameters (n, p). Thus, the PDF of T is given by

$$g_T(t) = \frac{p^n}{\Gamma(n)} e^{-pt} t^{n-1} \; ; t > 0, p > 0 \tag{6.1}$$

Therefore,

$$E\left(t^{-r}\right) = \frac{p^n}{\Gamma(n)} \int_0^{\infty} e^{-pt} t^{n-r-1}\, dt = p^r \frac{\Gamma(n-r)}{\Gamma(n)} \tag{6.2}$$

$$\therefore E\left(t^{-1}\right) = E\left(\frac{1}{t}\right) = \frac{p}{n-1}$$

and $E\left(t^{-2}\right) = E\left(\frac{1}{t^2}\right) = \frac{p^2}{(n-1)(n-2)}$ (6.3)

Thus, $V\left(\frac{1}{t}\right) = E\left(\frac{1}{t}\right)^2 - \left[E\left(\frac{1}{t}\right)\right]^2 = \frac{p^2}{(n-1)^2 (n-2)}$. (6.4)

We now obtain the relative efficiency of the estimators $\hat{p}_{KL}, \hat{p}_{EL}, \hat{p}_{SL}$ in case of both the priors.
We have,

$$\hat{p}_{ML} = \frac{n}{T} \Rightarrow V(\hat{p}_{ML}) = \frac{n^2 p^2}{(n-1)^2 (n-2)},$$

and $\hat{p}_{KL} = \frac{\sqrt{\beta_i(\beta_i - 1)}}{T} \Rightarrow V(\hat{p}_{KL}) = \frac{\beta_i(\beta_i - 1)p^2}{(n-1)^2 (n-2)},$

and $\hat{p}_{EL} = \frac{\beta_i - 1}{T} \Rightarrow V(\hat{p}_{EL}) = \frac{(\beta_i - 1)^2 p^2}{(n-1)^2 (n-2)},$

and $\hat{p}_{SL} = \frac{e^{\psi(\beta_i)}}{T} \Rightarrow V(\hat{p}_{SL}) = \frac{e^{2\psi(\beta_i)} p^2}{(n-1)^2 (n-2)},$

where $T, \beta_i\ (i = 1,2)$ are shape and rate parameters of gamma distribution defined in (3.3) and (3.5) respectively.

Thus, the efficiency of $\hat{p}_{EL}$ w.r.t. $\hat{p}_{KL}$ is given by

$$e_1 = \frac{V(\hat{p}_{KL})}{V(\hat{p}_{EL})} = \frac{\beta_i}{(\beta_i - 1)} > 1. \tag{6.5}$$

The efficiency of $\hat{p}_{EL}$ w.r.t. $\hat{p}_{SL}$ is given by

$$e_2 = \frac{V(\hat{p}_{SL})}{V(\hat{p}_{EL})} = \frac{e^{2\psi(\beta_i)}}{(\beta_i - 1)^2}. \tag{6.6}$$

The efficiency of $\hat{p}_{SL}$ w.r.t. $\hat{p}_{KL}$ is given by

$$e_3 = \frac{V(\hat{p}_{KL})}{V(\hat{p}_{SL})} = \frac{\beta_i(\beta_i - 1)}{e^{2\psi(\beta_i)}}. \tag{6.7}$$

7. Risk Functions of Bayes Estimators using Loss Functions K-LF, ELF and SLLF

Using K-LF, the risk function of $\hat{p}_{KL}$ using K-loss function is given by

$$R(\hat{p}_{KL}) = E[l(\hat{p}_{KL}, p)]$$
$$= p\left[\frac{\sqrt{\beta_i(\beta_i - 1)}}{p^2} E\left(\frac{1}{T}\right) - \frac{2}{p} + \frac{E(T)}{\sqrt{\beta_i(\beta_i - 1)}}\right]$$
$$= \frac{\sqrt{\beta_i(\beta_i - 1)}}{(n-1)} - 2 + \frac{n}{\sqrt{\beta_i(\beta_i - 1)}}. \tag{7.1}$$

Using ELF, the risk function of $\hat{p}_{EL}$ using entropy loss function is given by

$$R(\hat{p}_{EL}) = E[l(\hat{p}_{EL}, p)]$$
$$= \left[\frac{\beta_i - 1}{p} E\left(\frac{1}{T}\right) - \log(\beta_i - 1) + E[\log(pT)] - 1\right]$$
$$= \frac{\beta_i - 1}{n-1} - \log(\beta_i - 1) + \psi(n) - 1. \tag{7.2}$$

Using SLLF, the risk function of $\hat{p}_{SL}$ using entropy loss function is given by

$$R(\hat{p}_{SL}) = E[l(\hat{p}_{SL}, p)]$$
$$= \psi^2(\beta_i) - 2\psi(\beta_i)E(\log pT) + E(\log pT)^2$$
$$= \psi^2(\beta_i) - 2\psi(\beta_i)\psi(n) + \psi'(n) + \psi^2(n). \tag{7.3}$$

We observe from (7.1), (7.2) and (7.3) that the risk functions K-LF, ELF, and SLLF are constant w.r.t.*p*. Hence, $R(\hat{p}_{KL})$, $R(\hat{p}_{EL})$ and $R(\hat{p}_{SL})$ are minimax estimators for parameter p in case of Dagum distribution. [Lehmann (1983); Theorem 2.1, corollary 2.1; section 2, chapter 4, p. 249-250].

8. Prior Predictive Distribution under Mukherjee-Islam Prior

The prior predictive distribution under the Mukherjee-Islam prior is defined by

$$g(y) = a\,\alpha_1\,\sigma^{-\alpha_2} \int_0^\infty \frac{p^{\alpha_2}}{b^{a\,p}}\, y^{a\,p-1}\left[1 + \left(\frac{y}{b}\right)^a\right]^{-(p+1)} dp$$
$$= \frac{a\,\alpha_1\,\sigma^{-\alpha_2}}{y\left[1 + \left(\frac{y}{b}\right)^a\right]} \int_0^\infty p^{\alpha_1}\, e^{-p\,S}\, dp$$

$$= \frac{a\,\sigma^{-\alpha_2}}{y\left[1+\left(\frac{y}{b}\right)^a\right]}\frac{\alpha_2{}^2\Gamma(\alpha_1)}{S^{\alpha_1+1}}.$$

9. Prior Predictive Distribution under Extension of Jeffreys Prior

The prior predictive distribution under the extension of Jeffreys prior is defined by

$$g(y)=a\int_0^\infty p^{1-2m}\,y^{a\,p-1}\left[1+\left(\frac{y}{b}\right)^a\right]^{-(p+1)}dp$$

$$= \frac{a}{y\left[1+\left(\frac{y}{b}\right)^a\right]}\int_0^\infty p^{2-2m}\,e^{-p\,S}\,dp$$

$$= \frac{a\,\Gamma(2-2m)}{y\left[1+\left(\frac{y}{b}\right)^a\right]S^{2-2m}}$$

10. Posterior Predictive Distribution under Different Priors

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2 ... x_n)$ under the exponential prior is defined by

$$\phi_i(y\,|\,x)=\int_0^\infty f_D(y\,|\,p)\pi_i(p\,|\,x)dp$$

$$= \frac{a\,T^{\beta_i}}{y\left[1+\left(\frac{y}{b}\right)^a\right]\Gamma(\beta_i)}\int_0^\infty p^{\beta_i}e^{-T\,p}\left[1+\left(\frac{y}{b}\right)^{-a}\right]^{-p}dp$$

$$= \frac{a\,T^{\beta_i}}{y\left[1+\left(\frac{y}{b}\right)^a\right]\Gamma(\beta_i)}\int_0^\infty p^{\beta_i}e^{-p(T+S)}dp$$

$$= \frac{\beta_i}{T}\left(1+\frac{S}{T}\right)^{-(\beta_i+1)}\frac{a}{y\left[1+\left(\frac{y}{b}\right)^a\right]}.$$

where T and β_i $(i=1,2)$ are shape and rate parameters of gamma distribution defined in (3.3) and (3.5) respectively.

11. Data set : (Gupta and Kundu, 2009)

A real data set is considered for illustration of the proposed methodology. This data set represents the marks in Mathematics for 48 students in the slow pace program in the year 2003:

29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, 31.

	a	*b*	α_1=*1.2*	α_1=*3.0*	α_1=*3.5*
$\hat{p}_{KL}$	**3.0**	**1.49**	11.03563 (0.02160)	11.33146 (0.02343)	11.60454 (0.02628)
	4.4	**1.55**	6.98450 (0.02160)	7.17173 (0.02343)	7.34456 (0.02628)
$\hat{p}_{EL}$	**3.0**	**1.49**	10.92244 (0.01082)	11.21826 (0.01176)	11.49132 (0.01324)
	4.4	**1.55**	6.91286 (0.01082)	7.10009 (0.01176)	7.27291 (0.01324)
$\hat{p}_{SL}$	**3.0**	**1.49**	11.03641 (0.02149)	11.33222 (0.02329)	11.60528 (0.02611)
	4.4	**1.55**	6.98500 (0.02149)	7.17222 (0.02329)	7.34503 (0.02611)

Table 2: Posterior estimates $(\hat{p})$ risks $R(\hat{p})$ (in braces) under Mukherjee-Islam prior

	b	*a*	*m=0.5*	*m=1.3*	*m=3.0*
$\hat{p}_{KL}$	**3.0**	**1.49**	10.80807 (0.02116)	10.44397 (0.02235)	9.67024 (0.03368)
	4.4	**1.55**	6.84047 (0.02116)	6.61003 (0.02235)	6.12034 (0.03368)
$\hat{p}_{EL}$	**3.0**	**1.49**	10.69489 (0.01060)	10.33081 (0.01119)	9.55714 (0.01670)
	4.4	**1.55**	6.76884 (0.01060)	6.53842 (0.01119)	6.04876 (0.01670)
$\hat{p}_{SL}$	**3.0**	**1.49**	10.80887 (0.02105)	10.44479 (0.02223)	9.67114 (0.03342)
	4.4	**1.55**	6.84098 (0.02105)	6.61056 (0.02223)	6.12091 (0.03342)

Table 3: Posterior estimates $(\hat{p})$ risks $R(\hat{p})$ (in braces) under Mukherjee-Islam prior

12. Simulation Study

The simulation study was conducted in R-software to examine the performance of Bayes estimates for the scale (p) parameter of the Dagum distribution under Mukherjee-Islam prior and the extension of Jeffreys prior. The process is replicated 1000 times and the average of the results has been presented in the tables below. The *VGAM* package is used for the simulation study. We choose n=25, 50, 100 to represent different sample sizes. Tables 4 and 5 summarizes the results for different values of

p=10.95, 6.92; b =3.0, 4.4; a=1.49, 1.55. The hyper-parameter values are chosen as: α_1=(1.2, 3.0, 3.5); m=(0.5, 1.3, 3.0). These values are chosen using MLE values in R software. The estimators are obtained and their respective risks are computed for admissibility under different loss functions.

n		b	a	p	α_1=1.2	α_1=3.0	α_1=3.5
20	$\hat{p}_{KL}$	3.0	1.49	10.95	13.34204 (0.05453)	14.18861 (0.06474)	14.97003 (0.08001)
		4.4	1.55	6.92	6.34090 (0.05453)	6.74323 (0.06474)	7.11461 (0.08001)
	$\hat{p}_{EL}$	3.0	1.49	10.95	13.02050 (0.02742)	13.86683 (0.03287)	14.64806 (0.04122)
		4.4	1.55	6.92	6.18808 (0.02742)	6.59031 (0.03287)	6.96159 (0.04122)
	$\hat{p}_{SL}$	3.0	1.49	10.95	13.34733 (0.05377)	14.19358 (0.06370)	14.97475 (0.07851)
		4.4	1.55	6.92	6.34341 (0.05377)	6.74560 (0.06370)	7.11685 (0.07851)
50	$\hat{p}_{KL}$	3.0	1.49	10.95	11.86409 (0.02071)	12.16954 (0.02239)	12.45148 (0.02502)
		4.4	1.55	6.92	7.29153 (0.02071)	7.47925 (0.02239)	7.65253 (0.02502)
	$\hat{p}_{EL}$	3.0	1.49	10.95	11.74720 (0.01037)	12.05263 (0.01124)	12.33456 (0.01261)
		4.4	1.55	6.92	7.21969 (0.01037)	7.40740 (0.01124)	7.58068 (0.01261)
	$\hat{p}_{SL}$	3.0	1.49	10.95	11.86487 (0.02060)	12.17029 (0.02226)	12.45222 (0.02487)
		4.4	1.55	6.92	7.29201 (0.02060)	7.47972 (0.02226)	7.65299 (0.02487)
100	$\hat{p}_{KL}$	3.0	1.49	10.95	9.94729 (0.01018)	10.07596 (0.01060)	10.19474 (0.01128)
		4.4	1.55	6.92	6.71789 (0.01018)	6.80478 (0.01060)	6.88499 (0.01128)
	$\hat{p}_{KL}$	3.0	1.49	10.95	9.89792 (0.00509)	10.02660 (0.00531)	10.14537 (0.00565)
		4.4	1.55	6.92	6.68454 (0.00509)	6.77143 (0.00531)	6.85165 (0.00565)
	$\hat{p}_{SL}$	3.0	1.49	10.95	9.94745 (0.01015)	10.07613 (0.01057)	10.19490 (0.01125)
		4.4	1.55	6.92	6.71799 (0.01015)	6.80488 (0.01057)	6.88510 (0.01125)

Table 4: Posterior estimates $(\hat{p})$ and risks $R(\hat{p})$ under Mukherjee-Islam prior

n		b	a	p	m=0.5	m=1.3	m=3.0
20	$\hat{p}_{KL}$	3.0	1.49	10.95	12.69081 (0.05196)	11.64880 (0.05949)	9.43425 (0.14284)
		4.4	1.55	6.92	6.03140 (0.05196)	5.53617 (0.05949)	4.48369 (0.14284)
	$\hat{p}_{EL}$	3.0	1.49	10.95	12.36947 (0.02609)	11.32783 (0.02984)	9.11435 (0.06831)
		4.4	1.55	6.92	5.87868 (0.02609)	5.38363 (0.02984)	4.33166 (0.06831)
	$\hat{p}_{SL}$	3.0	1.49	10.95	12.69637 (0.05127)	11.65486 (0.05860)	9.44173 (0.13899)
		4.4	1.55	6.92	6.03404 (0.05127)	5.53905 (0.05860)	4.48725 (0.13899)
50	$\hat{p}_{KL}$	3.0	1.49	10.95	11.62914 (0.02031)	11.25321 (0.02140)	10.45435 (0.03177)
		4.4	1.55	6.92	7.14713 (0.02031)	6.91609 (0.02140)	6.42512 (0.03177)
	$\hat{p}_{EL}$	3.0	1.49	10.95	11.51226 (0.01017)	11.13635 (0.01071)	10.33754 (0.01576)
		4.4	1.55	6.92	7.07530 (0.01017)	6.84427 (0.01071)	6.35333 (0.01576)
	$\hat{p}_{SL}$	3.0	1.49	10.95	11.62993 (0.02020)	11.25403 (0.02128)	10.45523 (0.03154)
		4.4	1.55	6.92	7.14762 (0.02020)	6.91659 (0.02128)	6.42566 (0.03154)
100	$\hat{p}_{KL}$	3.0	1.49	10.95	9.84831 (0.01008)	9.68994 (0.01034)	9.35341 (0.01275)
		4.4	1.55	6.92	6.65103 (0.01008)	6.54407 (0.01034)	6.31680 (0.01275)
	$\hat{p}_{EL}$	3.0	1.49	10.95	9.79894 (0.00504)	9.64058 (0.00517)	9.30405 (0.00636)
		4.4	1.55	6.92	6.61769 (0.00504)	6.51074 (0.00517)	6.28346 (0.00636)
	$\hat{p}_{SL}$	3.0	1.49	10.95	9.84848 (0.01005)	9.69011 (0.01031)	9.35358 (0.01271)
		4.4	1.55	6.92	6.65114 (0.01005)	6.54419 (0.01031)	6.31692 (0.01271)

Table 5: Posterior estimates $(\hat{p})$ and risks $R(\hat{p})$ under extended Jeffreys prior

13. Results and Discussion

The results of the real data example and the simulation study are presented in tables 2 to 5 for different values of *n, a, b* and hyper-parameters. It is observed that

1. the risk values of the *p* under both Mukherjee-Islam prior and extension of Jeffreys prior are increasing as the hyper-parameter values increase.

2. the risk functions are constant w.r.t. *p*. i.e., there is no effect of increasing of the true values of the parameter *p* which implies that the Bayes estimators are minimax estimators for parameter *p* in case of Dagum distribution.
3. using ELF, the risk based on both priors is minimum and hence it is admissible for all *n*.
4. under K-LF and SLLF, the risks (although greater than that using ELF) based on both priors are almost same.
5. the risks are also seen decreasing for increasing sample size.
6.

Finally, from the results, we conclude that in situations involving estimation of parameter *p* of Dagum distribution, entropy loss function could be effectively employed instead of using a K-loss and squared log loss function owing to the least risk values among all the three loss functions.

Furthermore, for future attempt, other prior distributions can also be used for posterior analysis to see the impact of loss functions on them as well.

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