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ON BAYESIAN ESTIMATION OF GENERALIZED AUGMENTED INVERSE GAUSSIAN STRENGTH RELIABILITY OF A SYSTEM UNDER DIFFERENT LOSS FUNCTIONS

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Abstract

This article deals with the problem of Bayes and Maximum Likelihood Estimation (MLE) of generalized augmented strength reliability of the equipment under Augmentation Strategy Plan (ASP), where ASP is suggested for enhancing the strength of weaker / early failed equipment under three possible cases. It is assumed that the Inverse Gaussian stress (Y) is subjected to equipment having Inverse Gaussian strength (X) and are independent to each other. Assuming informative (gamma and inverted gamma) types of priors, the Bayes estimates of augmented strength reliability have been computed and compared with that of MLE on the basis of mean square errors (mse) and absolute biases. The posterior means under Squared Error Loss Function (SELF) as well as Linex Loss Function (LLF) are approximated by using Markov Chain Monte Carlo (MCMC). The mse and absolute biases are calculated with 1000 replications of the whole simulation process.

Key Words: Stress-Strength Reliability, Inverse Gaussian Distribution, MLE, Bayes Estimation, Gamma and Inverted Gamma Priors, MCMC Simulation, Metropolis-Hastings Algorithm.

1. Introduction

The Inverse Gaussian distribution (IGD) is a positively skewed and most commonly used life time distribution for time to event analysis and particularly considered where the initial failure rate is high. It is also considered as an alternative to the Weibull, Lognormal and similar distributions. Cox and Miller (1968) defined the Inverse Gaussian as a model in context of the first passage time in Brownian motion. Tweedie (1957a, 1957b) and Chhikara and Folks (1974, 1975, 1976, 1977) presented various real life applications of IG distribution and also Padgett (1981) described about its sampling inferences. According to Johnson et al. (1994), the IG distribution has many useful real life applications in various fields e.g. reliability and lifetime data analysis, theoretical physics to meteorology, sequential analysis and industrial quality control, business applications and so on. The IGD has an attractive property known as reproductive property which relates to the application of augmentation strategy plan (ASP). The probability density function of two-parameter IGD is given by

$$f_X(x)=\sqrt{\frac{\lambda}{2\pi x^3}}\ \exp-\left\{\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}; \quad x\geq 0,\mu,\lambda>0 \tag{1}$$

which, is represented by $IG(\mu,\lambda)$ with mean μ and variance μ^3/λ .

A number of researchers have proposed IGD as a life time distribution for life time data analysis for reliability and life testing as well as for survival analysis. Banerjee and Bhattacharyya (1979), Padgett and Wei (1979), Sherif and Smith (1980) and Sinha (1986) have attempted Bayesian estimation of parameters and reliability function of two-parameters IGD for different types of prior distributions. Padgett (1981) attempted the Bayesian estimation of reliability of two-parameter IGD by assuming Jeffrey prior and remarked that the choice of Jeffrey prior leads to an intractable posterior to estimate the reliability function. Latter on Howlader (1985) extended the result of Padgett (1981) and gave an approximate Bayes estimate of reliability function of two-parameter IGD using Jeffrey's non informative prior. Ahmad and Jaheen (1995) and Pandey and Bandyopadhyay (2013) have approximated the Bayes estimate of parameters and reliability function of two-parameter IGD with the choice of conjugate informative priors through Lindley and Gibbs sampler methods.

The problems of stress-strength reliability have been focused by many researchers over the last five to six decades for different choices of lifetime distributions. One may refer to some of latest literature works on system reliability like Sharma et al. (2015) and Sarhan et al. (2015) and references therein. Moreover, the two-parameter IGD have also been considered for assessment of stress-strength reliability for stochastic systems. Basu and Ebrahimi (1983) introduced a dynamic approach to model the time varying system reliability in the presence of stress process $X(t)$, which is faced by the system at time t and the strength process $Y(t)$. In the similar fashion Ebrahimi and Ramallingam (1993) and Basu and Lingham (2003) have also attempted the problem of modeling system reliability in the presence of stress and strength process with Brownian stress-strength model.

In general, the main focus of statistical reliability theory is to analyze the failure time data for reliability prediction and to aware its failure rate and also about some unknown characteristics. For any system, there exists two obvious characteristics i.e. either the system is reliable or unreliable. For obtaining the sufficient failure data of reliable system, the well developed accelerated life testing (see. Nelson 1982) is frequently used but so far there is no such strategy or methods available in the existing literatures on reliability theory for assessing the unreliable system. It is seen that many newly manufactured products or systems being failed at very early stage of its use and frequent failures occurs in used one due to poor quality of materials used. Since cost, time and manpower are involved in manufacturing the system, hence such products can not be wasted and can be reused by considering ASP. Thus, ASP (Chandra and Sen, 2014) is useful in enhancing the strength reliability and may be protected from unwanted early failure; hence enhanced strength of system may sustain to survive its usual life. Therefore such equipments may become in a reliable state in a hope to survive its usual life.

The idea of augmenting strength reliability of equipment under three possible cases was proposed by Alam and Roohi (2002) with assumption that the both stress and strength follow exponential life time distribution. Later on after one decade, Chandra and Sen (2014) attempted the augmentation problem for gamma distribution and named all the three cases as Augmentation Strategy Plans (ASP). Recently, Chandra and Rathaur (2015a) derived the augmented system strength reliability models by assuming that the stress and strength are independently and identically distributed as inverse Gaussian life distribution and they discovered that the ASP performs effetely in order to enhance the system reliability. Chandra and Rathaur (2015b, 2015c) tried the augmentation problem for coherent systems when the components are connected in series and parallel set up of connections for exponential and gamma lifetime distributions respectively.

In this paper we assume that the stress (Y) and the strength (X) are independent and identically distributed random variables which follow two parameter Inverse Gaussian (IG) distributions with probability density functions (pdf) of X (or Y) given in equation (1). The present work deals with the Bayesian estimation of the augmented strength reliability under generalized case of ASP for gamma and inverted gamma priors along with its classical counterpart MLE. Two different choices of loss functions i.e. symmetric (SELF) and asymmetric (LLF) have been considered for the Bayesian estimation and the comparison between both the methods of estimation (i.e. Bayes and ML) have been carried out on the basis of mean square errors (mse) and absolute biases through Markov Chain Monte Carlo methods of simulation.

The remaining part of article is organized as follows. In section 2, a generalized form of augmented strength reliability models under ASP is presented. In section 3, the ML estimators of generalized augmented strength reliability parameters under ASP are derived. In section 4, Bayes estimates of generalized augmented strength reliability parameters for both the loss functions are derived. In section 5, simulation study and discussions have been made on the basis of MCMC samples. Finally in section 6, the concluding remarks are given.

2. Augmented Strength Reliability for Generalized case of ASP

In this section, we introduce the augmented strength reliability model for the generalized case of ASP. Under the generalized case of ASP, the system strength is increased by adding 'n' identical components each having strength (X_i), which is 'm' times of the initial random common stress imposed on the equipment, is set to face the common stress (Y) which follows $IG(\mu,\lambda)$. Thus the increased strength of the equipment $Z_k=\sum_{i=1}^{n} X_i$ follows IGD with parameters $(nm\mu, n^2 m\lambda)$, where each X_i is independent and identically distributed random variable having $IG(m\mu, m\lambda)$, for $i=1,2,\dots,n.$ The probability density function of augmented strength (Z_k) random variable is given by

$$f_{Z_k}(z)=\sqrt{\frac{n^2 m\lambda}{2\pi\, z_k{}^3}}\ \exp-\left\{\frac{n^2 m\lambda(z_k-nm\mu)^2}{2n^2 m^2\mu^2 z_k}\right\};\ z_k\ge 0,\mu,\lambda,m>0,\ n>1 \quad (2)$$

where, 'm' is positive real number and 'n' is positive integer. From equation (2), the probability density functions of two special cases (case-I, II) of ASP can be obtained by substituting $k=1, n=1$ and $k=2, m=1$ respectively.

The augmented strength reliability $R_k = P(Z_k > Y)$ for the generalized case of ASP is defined as

$$R_k = \frac{n}{2}\sqrt{\frac{m\lambda}{2\pi}}\, e^{\frac{n\lambda}{\mu}}\left[\int_0^\infty erfc\left(\frac{\sqrt{\frac{\lambda}{z_k}}\left(-z_k+\mu\right)}{\sqrt{2}\,\mu}\right) z^{-3/2}\exp\left\{\frac{-\lambda z_k}{2m\mu^2}-\frac{n^2 m\lambda}{2z_k}\right\}\partial z_k \right.$$
$$\left. + e^{2\lambda/\mu}\int_0^\infty erfc\left(\frac{\sqrt{\frac{\lambda}{z_k}}\left(z_k+\mu\right)}{\sqrt{2}\,\mu}\right) z^{-3/2}\exp\left\{\frac{-\lambda z_k}{2m\mu^2}-\frac{n^2 m\lambda}{2z_k}\right\}\partial z_k \right] \quad (3)$$

where, $erfc(.)$ is complementary error function and it can also be defined in terms of error function. The error function equals the twice of the integral of a normalized Gaussian function between $0 \; and \; x/\sigma\sqrt{2}$, which is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt. \quad (4)$$

Thus the complementary error function is given as

$$erfc(x) = 1 - erf(x). \quad (5)$$

Equation (3) is the more generalized form (i.e. case-III) of augmented strength reliability model under ASP. The expressions for augmented strength reliability of other two special cases (case-I and II) can also be obtained by substituting $k=1, n=1$ and $k=2, m=1$ respectively in equation (3).

3. Maximum Likelihood Estimation of Generalized Augmented Strength Reliability

This section deals with the maximum likelihood estimation of parameters of generalized augmented strength $R_k = P(Z_k > Y)$, where, Z_k ; $k=1,2,3$ stands for case-I, II, III respectively, is the augmented random strength and Y being the common random stress imposed on the equipment are independent and identically distributed as Inverse Gaussian life time distribution with parameters μ and λ. Suppose $Z_k = \{Z_{k1}, Z_{k2}, \ldots, Z_{kn_1}\}$ and $Y = \{Y_1, Y_2, \ldots, Y_{n_2}\}$ being the two independent random samples of sizes n_1 and n_2 are drawn from the augmented inverse Gaussian strength and inverse Gaussian stress distributions respectively. Then likelihood function based on the observed random samples is defined as follows

$$L_k\left(z_k, y / \mu,\lambda\right) = n^{n_1}\, m^{\frac{n_1}{2}} \left(\frac{\lambda}{2\pi}\right)^{\frac{n_1+n_2}{2}} \prod_{i=1}^{n_1} z_{ki}^{\left(-3/2\right)} \prod_{j=1}^{n_2} y_j^{\left(-3/2\right)}$$

$$\exp\left[-\frac{\lambda}{2\mu^2}\left\{\left(\frac{n_1\,\bar{z}_k}{m} + n_2\bar{y}\right) - 2\mu\left(nn_1 + n_2\right) + \mu^2\left(n^2 m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right) + \sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\right\}\right] \tag{6}$$

The log likelihood equations with respect to μ and λ are given by

$$\frac{\partial \log L_k(data/\mu,\lambda)}{\partial\mu} = 0 \quad \text{and} \quad \frac{\partial \log L_k(data/\mu,\lambda)}{\partial\lambda} = 0 \tag{7}$$

The maximum likelihood estimators of μ and λ are obtained as

$$\hat{\mu} = \frac{(n_1\bar{z}_k/m) + n_2\bar{y}}{(nn_1 + n_2)} \tag{8}$$

$$\hat{\lambda} = (n_1 + n_2)\left[n^2 m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right) + \sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right) - \frac{(nn_1 + n_2)^2}{(n_1\bar{z}_k/m) + n_2\bar{y}}\right]^{-1} \tag{9}$$

Thus the maximum likelihood estimate of generalized augmented strength reliability can be obtained by substituting $\hat{\mu}$ and $\hat{\lambda}$ in place of μ and λ in the expression of R_k.

Remark: The MLE of augmented strength reliability $\left(R_k\right)$ for Cases-I, II and III under ASP can be obtained directly by substituting $k = n = 1;\ k = 2, m = 1;$ and $k = 3$ respectively in the expression of $\hat{R}_k$.

4. Bayesian Estimation of Generalized Augmenting Strength Reliability Models

In this section, we consider Bayes estimation of generalized augmenting strength reliability $R_k\left(k = 1,2,3\right)$ of ASP. In Bayesian paradigm, the model parameters are considered as random variables and a distribution is associated with them, called as prior distribution. There is no hard and fast rule available in the literature for making a choice of prior distributions and thus it is also a tedious task. The general ideology behind the choice of such prior is subjective matter, depends on personal belief and based on past experiences or the available data from the past experiments for same type of situation in the real life. As per our literature study, no researcher have reported any real life example behind the choice of priors(s). Berger (1985) suggested that if the proper prior information is available, it is better to use the informative prior(s) than the non-informative prior(s), which are combined to the current information to update the belief regarding a particular characteristic of the data. In this study, we consider μ and λ are independent random variables having informative (gamma and inverted gamma priors) types of priors. The intension behind choosing two informative priors is just to know which of the prior distribution is a better choice with greater impacts for Bayesian analysis.

4.1 Assuming Gamma prior distribution

In this study, considering μ and λ as independent random variables having gamma informative type of priors i.e., $\mu \sim G(a_1, b_1)$ and $\lambda \sim G(c_1, d_1)$ the joint prior probability density function of μ and λ is given by

$$g_1(\mu,\lambda) \propto \mu^{a_1-1}\lambda^{c_1-1}\exp\left\{-\left(b_1\mu + d_1\lambda\right)\right\} \; ;\mu,\lambda > 0 \; ;a_1,\, b_1,\, c_1,\, d_1 > 0 \tag{10}$$

The hyper-parameters a_1, b_1, c_1 and d_1 of prior density function are assumed to be known and are chosen in such a way to reflect the prior belief about the unknown parameters. Thus the joint posterior probability distribution of random variables μ and λ for generalized case of ASP is obtained by combining both the likelihood function $L_k(data / \mu, \lambda)$ and the joint prior probability density $g_1(\mu,\lambda)$. The joint posterior probability distribution under gamma prior is given as

$$\Pi_{k1}\left(\mu,\, \lambda / data\right) \propto \mu^{a_1-1}\lambda^{\frac{n_1+n_2}{2}+c_1-1}\exp\Bigg[-b_1\mu - d_1\lambda - \frac{\lambda}{2\mu^2}\Bigg\{\left(\frac{n_1\bar{z}_k}{m}+n_2\bar{y}\right) - 2\mu\left(nn_1+n_2\right)+\mu^2\left(n^2m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right)+\sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\Bigg\}\Bigg]. \tag{11}$$

The Bayes estimator of generalized augmented strength reliability under squared error loss function is given as

$$\hat{R}_k^{self} = \int_{(\mu,\lambda)} R_k \; \Pi_{k1}\left(\mu,\lambda / \mathbf{data}\right)\partial\mu\,\partial\lambda \tag{12}$$

$$\hat{R}_k^{self} \propto \int_0^\infty\int_0^\infty R_k\;\mu^{a_1-1}\lambda^{\frac{n_1+n_2}{2}+c_1-1}\exp\Bigg[-b_1\mu - d_1\lambda - \frac{\lambda}{2\mu^2}\Bigg\{\left(\frac{n_1\bar{z}_k}{m}+n_2\bar{y}\right) - 2\mu\left(nn_1+n_2\right)+\mu^2\left(n^2m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right)+\sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\Bigg\}\Bigg]\partial\mu\,\partial\lambda \tag{13}$$

where, R_k is the augmented strength reliability model for generalized case of ASP, given in equation (3). Under Linex loss function, the Bayes estimate of augmented strength reliability $R_k\left(k=1,2,3\right)$ for generalized case of ASP is given as

$$\hat{R}_k^{llf} = \frac{-1}{p}\ln\left[\int_0^\infty\int_0^\infty e^{-pR_k}\;\Pi_{k1}\left(\mu,\lambda / data\right)\partial\mu\,\partial\lambda\right] \tag{14}$$

$$\propto \frac{-1}{p}\ln\left[\begin{array}{l}\int_0^\infty\int_0^\infty e^{-pR_k}\,\mu^{a_1-1}\,\lambda^{\frac{n_1+n_2}{2}+c_1-1}\,\exp\left[-b_1\mu-d_1\lambda-\frac{\lambda}{2\mu^2}\left\{\left(\frac{n_1\,\overline{z}_k}{m}+n_2\overline{y}\right)\right.\right.\\ \left.\left.\left.-2\mu\left(nn_1+n_2\right)+\mu^2\left(n^2m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right)+\sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\right\}\right]\partial\mu\,\partial\lambda\end{array}\right]. \quad (15)$$

The above expressions of Bayes estimators of augmented strength reliability under SELF and LLF obtained in (13) and (15) are not in explicit forms and therefore cannot be evaluated manually. As an alternatively, numerical approximations through Markov-Chain Monte-Carlo (MCMC) have been done to evaluate these expressions.

4.2 Assuming Inverted Gamma prior distribution

We assume that the model parameters μ and λ are independent random variables which follow inverted gamma prior distribution with hyper-parameters (a_2,b_2) and (c_2,d_2) respectively. Thus the joint prior probability density function of μ and λ is defined as follows

$$g_2(\mu,\lambda)\propto\ \mu^{-a_2-1}\,\lambda^{-c_2-1}\exp\left\{-\left(\frac{b_2}{\mu}+\frac{d_2}{\lambda}\right)\right\};\mu,\lambda>0;a_2,b_2,c_2,d_2>0. \quad (16)$$

Thus the joint posterior distribution of μ and λ can be defined as

$$\begin{array}{l}\prod_{k2}\left(\mu,\,\lambda/\,data\right)\propto\ \mu^{-a_2-1}\,\lambda^{\frac{n_1+n_2}{2}-c_2-1}\exp\left[-\frac{b_2}{\mu}-\frac{d_2}{\lambda}-\frac{\lambda}{2\mu^2}\left\{\left(\frac{n_1\,\overline{z}_k}{m}+n_2\overline{y}\right)\right.\right.\\ \left.\left.\qquad-2\mu\left(nn_1+n_2\right)+\mu^2\left(n^2m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right)+\sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\right\}\right].\end{array} \quad (17)$$

Thus the Bayes estimate of generalized augmented strength reliability under ASP for square error loss function is defined by

$$\hat{R}_k^{\,self}=\int_{(\mu,\lambda)}R_k\,\prod_{k2}\left(\mu,\lambda/\,\mathbf{data}\right)\partial\mu\,\partial\lambda \quad (18)$$

$$\begin{array}{l}\hat{R}_k^{\,self}\propto\int_0^\infty\int_0^\infty R_k\,\mu^{-a_2-1}\,\lambda^{\frac{n_1+n_2}{2}-c_2-1}\,\exp\left[-\frac{b_2}{\mu}-\frac{d_2}{\lambda}-\frac{\lambda}{2\mu^2}\left\{\left(\frac{n_1\,\overline{z}_k}{m}+n_2\overline{y}\right)\right.\right.\\ \left.\left.\qquad-2\mu\left(nn_1+n_2\right)+\mu^2\left(n^2m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right)+\sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\right\}\right]\partial\mu\,\partial\lambda.\end{array} \quad (19)$$

Under Linex Loss function, the Bayes estimator ($\hat{R}_k^{\,llf}$) of generalized augmented strength reliability of ASP is given as

$$\hat{R}_k^{\,llf}=\frac{-1}{p}\ln\left[\int_{(\mu,\,\lambda)}e^{-p.R_k}\,\prod_{k2}\left(\mu,\lambda/\,\mathbf{data}\right)\partial\mu\,\partial\lambda\right] \quad (20)$$

$$\propto \frac{-1}{p}\ln\left[\begin{array}{c}\int_0^\infty\int_0^\infty e^{-pR_k}\,\mu^{-a_2-1}\,\lambda^{\frac{n_1+n_2}{2}-c_2-1}\;\exp\left[-\frac{b_2}{\mu}-\frac{d_2}{\lambda}-\frac{\lambda}{2\mu^2}\left\{\left(\frac{n_1\,\overline{z}_k}{m}+n_2\overline{y}\right)\right.\right.\\ \left.\left.-2\mu\left(nn_1+n_2\right)+\mu^2\left(n^2m\sum_{i=1}^{n_1}\left(\frac{1}{z_{ki}}\right)+\sum_{j=1}^{n_2}\left(\frac{1}{y_j}\right)\right)\right\}\right]\partial\mu\,\partial\lambda\end{array}\right]. \tag{21}$$

Here it is observed that in each of the selected prior cases, the joint posterior densities are not in any known distributional form and therefore finding the posterior expectations of generalized augmented strength reliability (R_k) is also complicated. Thus the analytical evaluation is impossible. In this situation, the Markov Chain Monte Carlo (MCMC) sampling method can be used to approximate these integrals (see; Brook, 1998). In order to approximate the above integrals for finding the Bayes estimates; we have used the Metropolis-Hastings (MH) algorithm to sample from the marginal posterior distributions. The following are the methodology for carrying out the approximations of posterior mean of generalized augmented strength reliability through the M-H algorithm.

Hence, the Bayes estimator under SELF ($\hat{R}_k^{self}$) of R_k under this sampling procedure is given by

$$\hat{R}_k^{self} = \frac{1}{N}\sum_{i=1}^{N}[R_k]_{\mu=\mu_i;\lambda=\lambda_i}$$

and the Bayes estimator $\hat{R}_k^{llf}$ of R_k under LLF is given as

$$\hat{R}_k^{llf} = \frac{1}{N}\left[\frac{1}{p}\ln\left(\sum_{i=1}^{N}e^{-p[R_k]_{\mu=\mu_i;\lambda=\lambda_i}}\right)\right]$$

where, $R_k(k=1,2,3)$ is the generalized augmented strength reliability model for general case of ASP and μ_i and λ_i $(i=1,2,...,N)$ are the random samples drawn from the marginal posterior distributions of μ and λ respectively through M-H algorithm.

5. Simulation Study and Discussion

In this section, the validation and comparison of the different methods of estimators (ML and Bayes) has been done through simulation study for different combinations of sample sizes and stress-strength reliability parameters with 1000 replications. The Bayes estimators of generalized augmented strength reliability for two different loss functions (i.e. SELF and LLF) under gamma and inverted gamma priors have been compared with that of ML estimators through their mean square errors and absolute biases. It may be noticed that the expressions of posterior expectations for both the priors are not in closed form, thus to find out numerical evaluation, MCMC technique viz. Metropolis-Hastings algorithm (Hasting, 1970) has been used for drawing the sample from the arbitrary posterior distribution. For generating a random sample of size N (say) from a density function $f(.)$ through Metropolis-Hastings algorithm, the following steps are given:

Step 1: set j=1 and establish starting value for the parameter $\theta^{(0)}$ such that $f\left(\theta^{(0)}\right)>0$.

Step 2: Draw a 'candidate' parameter θ^c from a proposal density say $\alpha(.)$.
Step 3: Compute the ratio

$$R = \frac{f(\theta^c)\,\alpha(\theta^{j-1}/\theta^c)}{f(\theta^{j-1})\,\alpha(\theta^c/\theta^{j-1})}$$

Step 4: Generate 'U' uniform variate on range 0 and 1 i.e. $u \sim U(0,1)$.
Step 5: If $u \le \min(1,R)$, accept the candidate point with probability $\min(1,R)$, otherwise set

$\theta^j = \theta^{j-1}$.

Step 6: Repeat steps 2-5 for all $j = 1,2,...,N$ and obtained $\theta^{(1)},\theta^{(2)},...,\theta^{(N)}$.

To evaluate the Bayes estimate of augmented strength reliability model five thousand random samples have been drawn by following the above steps of Metropolis-Hastings algorithm by considering asymptotic normal distribution as proposal distribution.

The following given tables 1, 2, 3 and 4 contain the average estimates, MSE and absolute biases for MLE and Bayes estimates for gamma as well as inverted gamma prior under each of SELF and LLF of generalized augmented strength reliability under ASP for different values of stress-strength reliability parameters μ, λ, n and m and the sample sizes (n_1, n_2) by fixing the hyper-parameters $a_1 = a_2 = 0.50;$ $b_1 = b_2 = 0.25;$ $c_1 = c_2 = 0.35;$ $d_1 = d_2 = 0.75$. The following observations are made based on the given tables.

- ∗ In Table 1, the effect of variation of $\mu(2.5, 4.5, 6.5)$ have been presented for the fixed value of other model parameters $\lambda = 2.5; m = n = 2$. It is observed from the table that the Bayes estimate of augmented strength reliability for generalized case of ASP gives better result than that of MLE on the basis of MSE and absolute biases. In comparison between the Bayes estimate for both the priors, it is noticed that the Bayes estimate for gamma prior under square error loss function has the minimum mean square error and absolute biases. It may also be noticed that the mean square errors and absolute biases gradually decrease for the increasing values of sample sizes (n_1, n_2) .
- ∗ The variation of $\lambda(1.5, 4.5)$ has been reported by fixing rest model parameters $\mu = 3.5; m = n = 2$ in Table 2. It may be noticed from the table that the true augmented strength reliability increases with increasing value of λ. Bayes estimates for gamma prior under SELF and LLF gives the more precise estimate than that of other estimators. The pattern of MSE and absolute biases are decreasing in nature for increasing values of sample sizes. Overall, Bayes estimates perform better in comparison with MLE on the basis of mean square errors and absolute biases.
- ∗ In Table 3, the variation of $n(2, 4, 8)$ have been shown by fixing other model parameters $\mu = 3.5;\ \lambda = 2.5;\ m = 2$ and it is observed that the true generalized

augmented strength reliability increases with increasing values of n (number of added components), i.e. the strength reliability is enhanced by adding more components. It may be observed from the table that the Bayes estimators gives better result than MLE for small value of $n(2)$ but for large $n(4,8)$ the MLE gives better result than that of Bayes estimators.

* Table 4 presents the variation of $m(5,7)$ by keeping other parameters constant $\mu = 3.5;\ \lambda = 2.5;\ n = 2$ and it is observed that the augmented strength reliability approaches 99% for higher values of m. It is also noticed that the Bayes estimate for gamma prior under SELF performs better among other estimators for small value of $m(5)$ but for higher values of $m(7)$, the MLE performs better. It may also be noticed that the mean square errors and absolute biases gradually decrease for increasing values of sample sizes.

μ = 2.5, R = 0.9154851						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.92752	0.92233	0.92192	0.92501	0.92462
	MSE	0.00074	0.00054	0.00055	0.00060	0.00061
	Abs. bias	0.00249	0.00271	0.00311	0.00002	0.00041
(10,30)	Avg.	0.91648	0.91393	0.91366	0.91564	0.91538
	MSE	0.00050	0.00043	0.00043	0.00044	0.00044
	Abs. bias	0.00099	0.00156	0.00183	0.00015	0.00010
(20,30)	Avg.	0.92875	0.92615	0.92598	0.92756	0.92739
	MSE	0.00031	0.00027	0.00028	0.00028	0.00028
	Abs. bias	0.00155	0.00105	0.00122	0.00036	0.00019
(30,20)	Avg.	0.92594	0.92361	0.92345	0.92481	0.92466
	MSE	0.00029	0.00026	0.00026	0.00027	0.00027
	Abs. bias	0.00087	0.00147	0.00162	0.00026	0.00041
(50,50)	Avg.	0.92640	0.92494	0.92486	0.92572	0.92564
	MSE	0.00015	0.00014	0.00014	0.00014	0.00014
	Abs. bias	0.00085	0.00060	0.00069	0.00017	0.00009
μ = 4.5, R = 0.8859389						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.88839	0.89667	0.89624	0.89300	0.89251
	MSE	0.00106	0.00068	0.00068	0.00083	0.00083
	Abs. bias	0.00126	0.00955	0.00912	0.00587	0.00538
(10,30)	Avg.	0.89516	0.89901	0.89873	0.89740	0.89710
	MSE	0.00067	0.00052	0.00052	0.00058	0.00059
	Abs. bias	0.00189	0.00574	0.00546	0.00413	0.00383
(20,30)	Avg.	0.88720	0.89035	0.89013	0.88929	0.88906
	MSE	0.00050	0.00041	0.00041	0.00045	0.00045
	Abs. bias	0.00126	0.00441	0.00419	0.00335	0.00312

(30,20)	Avg.	0.88497	0.88796	0.88775	0.88683	0.88662
	Mse	0.00046	0.00039	0.00039	0.00042	0.00042
	Abs. bias	0.00152	0.00451	0.00430	0.00338	0.00317
(50,50)	Avg.	0.88591	0.88741	0.88730	0.88684	0.88673
	MSE	0.00023	0.00021	0.00021	0.00022	0.00022
	Abs. bias	0.00053	0.00097	0.00086	0.00040	0.00029
$\mu = 6.5$, R = 0.8585458						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.86145	0.88125	0.88081	0.87110	0.87054
	MSE	0.00148	0.00121	0.00120	0.00125	0.00124
	Abs. bias	0.00291	0.02270	0.02227	0.01256	0.01199
(10,30)	Avg.	0.86839	0.87901	0.87872	0.87370	0.87336
	MSE	0.00080	0.00071	0.00071	0.00074	0.00073
	Abs. bias	0.00345	0.01407	0.01378	0.00876	0.00842
(20,30)	Avg.	0.87368	0.88146	0.88124	0.87759	0.87734
	MSE	0.00054	0.00049	0.00048	0.00050	0.00050
	Abs. bias	0.00244	0.01022	0.01000	0.00635	0.00610
(30,20)	Avg.	0.86686	0.87451	0.87430	0.87071	0.87048
	MSE	0.00054	0.00046	0.00046	0.00049	0.00049
	Abs. bias	0.00083	0.00848	0.00827	0.00468	0.00445
(50,50)	Avg.	0.86143	0.86560	0.86548	0.86356	0.86343
	MSE	0.00027	0.00026	0.00026	0.00026	0.00026
	Abs. bias	0.00185	0.00602	0.00590	0.00398	0.00385

Table 1: AVG, MSE and Abs. Bias for estimates of R_k for variation of μ when $\lambda = 2.5; m = n = 2$.

$\lambda = 1.5$, R = 0.8795406						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.88260	0.89075	0.89030	0.88850	0.88801
	MSE	0.00119	0.00084	0.00084	0.00098	0.00098
	Abs. bias	0.00306	0.01121	0.01076	0.00896	0.00847
(10,30)	Avg.	0.87138	0.87683	0.87651	0.87594	0.87560
	MSE	0.00078	0.00064	0.00064	0.00071	0.00070
	Abs. bias	0.00189	0.00734	0.00702	0.00645	0.00611
(20,30)	Avg.	0.86535	0.86975	0.86949	0.86898	0.86871
	MSE	0.00057	0.00048	0.00048	0.00051	0.00051
	Abs. bias	0.00093	0.00533	0.00508	0.00456	0.00430
(30,20)	Avg.	0.87484	0.87830	0.87808	0.87772	0.87749
	MSE	0.00048	0.00040	0.00040	0.00044	0.00044
	Abs. bias	0.00060	0.00406	0.00384	0.00347	0.00324
(50,50)	Avg.	0.87610	0.87793	0.87782	0.87775	0.87763
	MSE	0.00023	0.00021	0.00021	0.00022	0.00022
	Abs. bias	0.00166	0.00349	0.00338	0.00330	0.00319

λ = 4.5, R = 0.9416981						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.94355	0.93470	0.93435	0.93889	0.93856
	MSE	0.00053	0.00043	0.00044	0.00046	0.00047
	Abs. bias	0.00186	0.00699	0.00735	0.00281	0.00313
(10,30)	Avg.	0.93962	0.93486	0.93465	0.93725	0.93705
	Mse	0.00035	0.00031	0.00031	0.00032	0.00032
	Abs. bias	0.00109	0.00367	0.00388	0.00129	0.00148
(20,30)	Avg.	0.94012	0.93598	0.93583	0.93807	0.93793
	MSE	0.00026	0.00025	0.00025	0.00025	0.00025
	Abs. bias	0.00018	0.00396	0.00411	0.00187	0.00201
(30,20)	Avg.	0.94204	0.93777	0.93764	0.93984	0.93972
	MSE	0.00022	0.00021	0.00022	0.00022	0.00022
	Abs. bias	0.00044	0.00383	0.00396	0.00176	0.00188
(50,50)	Avg.	0.94351	0.94111	0.94105	0.94235	0.94229
	MSE	0.00012	0.00011	0.00011	0.00011	0.00011
	Abs. bias	0.00028	0.00212	0.00218	0.00088	0.00094

Table 2: AVG, MSE and Abs. Bias for estimates of R_k for variation of λ when $\mu = 3.5; m = n = 2$.

n = 2, R = 0.8985011						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.906292	0.907458	0.907018	0.907341	0.906879
	MSE	0.000999	0.000641	0.000642	0.000780	0.000784
	Abs. bias	0.003801	0.004968	0.004527	0.004850	0.004389
(10,30)	Avg.	0.896346	0.897823	0.897517	0.897756	0.897443
	MSE	0.000715	0.000558	0.000560	0.000612	0.000615
	Abs. bias	0.000461	0.001938	0.001632	0.001871	0.001558
(20,30)	Avg.	0.906355	0.906816	0.906618	0.907002	0.906803
	MSE	0.000396	0.000328	0.000329	0.000354	0.000355
	Abs. bias	0.001662	0.002122	0.001924	0.002308	0.002110
(30,20)	Avg.	0.901272	0.901793	0.901612	0.901935	0.901754
	MSE	0.000380	0.000315	0.000316	0.000340	0.000341
	Abs. bias	0.000512	0.001034	0.000852	0.001175	0.000994
(50,50)	Avg.	0.898563	0.898759	0.898656	0.898894	0.898790
	MSE	0.000209	0.000190	0.000191	0.000198	0.000198
	Abs. bias	0.000062	0.000258	0.000154	0.000393	0.000289
n = 4, R = 0.9832689						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.983279	0.977953	0.977877	0.979624	0.979550
	MSE	0.000092	0.000107	0.000109	0.000103	0.000104

	Abs. bias	0.000359	0.004967	0.005043	0.003296	0.003369
(10,30)	Avg.	0.983989	0.981101	0.981057	0.981996	0.981955
	MSE	0.000057	0.000062	0.000062	0.000060	0.000060
	Abs. bias	0.000343	0.002545	0.002589	0.001650	0.001691
(20,30)	Avg.	0.985159	0.982782	0.982756	0.983589	0.983565
	MSE	0.000033	0.000038	0.000039	0.000036	0.000036
	Abs. bias	0.000297	0.002079	0.002105	0.001272	0.001296
(30,20)	Avg.	0.984799	0.982527	0.982506	0.983268	0.983249
	MSE	0.000030	0.000034	0.000034	0.000033	0.000033
	Abs. bias	0.000234	0.002037	0.002058	0.001296	0.001316
(50,50)	Avg.	0.983049	0.981854	0.981843	0.982266	0.982255
	MSE	0.000019	0.000020	0.000021	0.000020	0.000020
	Abs. bias	0.000220	0.001415	0.001426	0.001003	0.001014
n = 8, R = 0.9991268						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.9990054	0.9973309	0.9973261	0.9978666	0.9978629
	MSE	0.0000013	0.0000064	0.0000065	0.0000043	0.0000044
	Abs. bias	0.0001214	0.0017960	0.0018007	0.0012602	0.0012639
(10,30)	Avg.	0.9989527	0.9981277	0.9981258	0.9983415	0.9983399
	MSE	0.0000009	0.0000027	0.0000028	0.0000022	0.0000022
	Abs. bias	0.0001576	0.0009826	0.0009844	0.0007687	0.0007703
(20,30)	Avg.	0.9991059	0.9985328	0.9985320	0.9986928	0.9986921
	MSE	0.0000005	0.0000014	0.0000014	0.0000011	0.0000011
	Abs. bias	0.0001310	0.0007041	0.0007049	0.0005441	0.0005448
(30,20)	Avg.	0.9990842	0.9985520	0.9985514	0.9987098	0.9987093
	MSE	0.0000004	0.0000010	0.0000010	0.0000007	0.0000007
	Abs. bias	0.0000580	0.0005902	0.0005908	0.0004324	0.0004329
(50,50)	Avg.	0.9991221	0.9988631	0.9988629	0.9989387	0.9989385
	MSE	0.0000002	0.0000003	0.0000003	0.0000003	0.0000003
	Abs. bias	0.0000189	0.0002779	0.0002781	0.0002023	0.0002025

Table 3: AVG, MSE and Abs. Bias for estimates of R_k for variation of n when $\mu = 3.5$, $\lambda = 2.5; m = 2$.

m = 5, R = 0.9808704						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.981476	0.977237	0.977140	0.978389	0.978288
	MSE	0.000125	0.000112	0.000114	0.000122	0.000124
	Abs. bias	0.000605	0.003633	0.003731	0.002481	0.002582
(10,30)	Avg.	0.978836	0.976402	0.976339	0.977115	0.977052
	MSE	0.000094	0.000091	0.000092	0.000093	0.000094
	Abs. bias	0.000152	0.002586	0.002649	0.001874	0.001937
(20,30)	Avg.	0.979773	0.977745	0.977703	0.978387	0.978347
	MSE	0.000063	0.000064	0.000065	0.000064	0.000064

	Abs. bias	0.000210	0.002237	0.002279	0.001596	0.001636
(30,20)	Avg. MSE Abs. bias	0.982698 0.000048 0.000142	0.980650 0.000049 0.001905	0.980618 0.000049 0.001937	0.981282 0.000049 0.001273	0.981251 0.000049 0.001305
(50,50)	Avg. MSE Abs. bias	0.980373 0.000027 0.000059	0.979346 0.000028 0.000968	0.979330 0.000028 0.000984	0.979686 0.000027 0.000628	0.979670 0.000027 0.000644
m = 7, R = 0.99035						
(n_1, n_2)	Statistic	MLE	Gamma prior		Inverted gamma prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg. MSE Abs. bias	0.990303 0.000047 0.000047	0.986837 0.000057 0.003513	0.986789 0.000058 0.003561	0.987640 0.000058 0.002710	0.987589 0.000059 0.002761
(10,30)	Avg. MSE Abs. bias	0.991388 0.000027 0.000173	0.989266 0.000034 0.002295	0.989241 0.000034 0.002320	0.989791 0.000033 0.001769	0.989766 0.000033 0.001794
(20,30)	Avg. MSE Abs. bias	0.990084 0.000023 0.000376	0.988534 0.000027 0.001926	0.988517 0.000027 0.001943	0.988941 0.000026 0.001519	0.988924 0.000026 0.001536
(30,20)	Avg. MSE Abs. bias	0.989655 0.000019 0.000004	0.988081 0.000022 0.001578	0.988066 0.000022 0.001593	0.988497 0.000022 0.001162	0.988482 0.000022 0.001177
(50,50)	Avg. MSE Abs. bias	0.991415 0.000009 0.000096	0.990602 0.000010 0.000909	0.990596 0.000010 0.000915	0.990845 0.000010 0.000666	0.990839 0.000010 0.000672

Table 4: AVG, MSE and Abs. Bias for estimates of R_k for variation of m when $\mu = 3.5$, $\lambda = 2.5; n = 2$.

6. Concluding Remarks

In this article, Bayesian estimation of generalized augmenting strength reliability parameters of inverse Gaussian distribution under ASP based on independent gamma and inverted gamma family of informative priors using symmetric and asymmetric loss functions is studied. The Bayes estimators augmenting strength reliability are compared with that of MLE. A simulation study is carried out to study the behavior of proposed estimators of augmenting strength reliability under various combinations of sample sizes (n_1, n_2) and model parameters including hyper parameters.

The simulation results given in tables suggest that the performance of the proposed Bayes estimates of augmented strength reliability based on informative priors perform better than ML estimates. It is also observed that the strength reliability of the system get enhanced by adding some desired level of components. Thus the ASP is suggestive for practical purposes in order to enhance the strength of weaker or failed systems.

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References

1. Alam, S. N. and Roohi (2002). On Augmenting Exponential Strength-Reliability, IAPQR Transactions, 27, p. 111-117.
2. Ahmad, K. E. and Jaheen, Z. F. (1995). Approximate Bayes estimators applied to the inverse Gaussian lifetime model, Computers & Mathematics with Applications, 29(12), p. 39-47.
3. Banerjee, A. K. and Bhattacharyya, G. K. (1979). Bayesian results for the inverse Gaussian distribution with an application, Technometrics, 21(2), p. 247-251.
4. Basu, A. P., and Ebrahimi, N. (1983). On the reliability of stochastic systems, Statistics & Probability Letters, 1(5), p. 265-267.
5. Basu, S. and Lingham, R. T. (2003). Bayesian estimation of system reliability in Brownian stress-strength models, Annals of the Institute of Statistical Mathematics, 55(1), p. 7-19.
6. Berger J.O. (1985). Statistical Decision Theory and Bayesian Analysis, Springer-Verlag: New York.
7. Brooks, S. (1998). Markov chain Monte Carlo method and its application, Journal of the royal statistical society: series D (the Statistician), 47(1), p. 69-100.
8. Chandra, N. and Sen, S. (2014). Augmented strength reliability of equipment under gamma distribution, J of Statist. Theory and Applications, 13, p. 212-221.
9. Chandra, N. and Rathaur, V. K. (2015a). Augmented strategy plans for enhancing strength reliability of an equipment under inverse gaussian distribution, J. Math. Engg. Sci. Aerospace: Special Issue on Reliability and Dependability Modeling Analysis for Complex Systems, 6, p. 233-243.
10. Chandra, N. and Rathaur, V. K. (2015b). Augmenting exponential stress-strength reliability for a coherent system. Proceedings of National Seminar on Statistical Methods and Data Analysis, published by Abhiruchi Prakashana, Mysore. p. 25-34.
11. Chandra, N. and Rathaur, V.K. (2015c). Augmented gamma strength reliability models for series and parallel coherent system, Proceedings of National Conference on Emerging Trends in Statistical Research: Issue and Challenges, Narosa Publication, New Delhi, India, p. 43-54.
12. Chhikara R.S. and Folks J. L. (1974). Estimation of the Inverse Gaussian distribution function, J. Amer. Statis. Assoc. 69, p. 250-254.
13. Chhikara R.S. and Folks J. L. (1975). Statistical distributions related to the inverse Gaussian, Communications in Statistics 4, 1081-1091.
14. Chhikara R.S. and Folks J. L. (1976). Optimum test procedures for the mean of first Passage time distributions in Brownian motion with positive drift, Technometrics, 18, p. 189-193.

15. Chhikara, R.S. and Folks J. L. (1977). The Inverse Gaussian distribution as a lifetime model, Technometrics, 19(4), p. 461-468.
16. Cox, D. R. and Miller, H. D. (1977). The Theory Of Stochastic Processes (Vol. 134), CRC Press.
17. Ebrahimi, N. and Ramallingam, T. (1993). Estimation of system reliability in Brownian stress-strength models based on sample paths, Annals of the Institute of Statistical Mathematics, 45(1), p. 9-19.
18. Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications, Biometrika, 57(1), p. 97-109.
19. Howlader, H. A. (1985). Approximate Bayes estimation of reliability of two parameter inverse Gaussian distribution, Communications in Statistics-Theory and Methods, 14(4), p. 937-946.
20. Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). Continuous Univariate Distributions (2nd ed.), Vol. 1, John Wiley & Sons, New York, p. 163.
21. Nelson, W.B.(1982). Applied Life Data Analysis. John Wiley & Sons, New York.
22. Padgett, W. J. and Wei, L. J. (1979). Estimation for the three-parameter inverse Gaussian distribution, Communications in Statistics-Theory and Methods, 8(2), p. 129-137.
23. Padgett, W. J. (1981). Bayes estimation of reliability for the inverse Gaussian model, IEEE Transactions on Reliability, 30(4), p. 384-385.
24. Pandey, B. N., and Bandyopadhyay, P. (2013). Bayesian estimation of inverse Gaussian distribution, Int. J. Agricult. Stat. Sci., 9(2), p. 373-386.
25. Sarhan, A. M., Smith, B. and Hamilton, D.C. (2015). Estimation of P(Y <X) for a two-parameter bathtub shaped failure rate distribution, I. J. of Statist. Prob., 4, p. 33-45.
26. Sharma,V.K., Singh, S.K., Singh, U. and Agiwal, V. (2015). The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data, J. of Indust. and Product. Engg., 32, p. 162–173.
27. Sherif, Y. S., and Smith, M. L. (1980). First-passage time distribution of Brownian motion as a reliability model, Reliability, IEEE Transactions on, 29(5), p. 425-426.
28. Sinha, S. K. (1986). Bayesian estimation of the reliability function of the inverse Gaussian distribution, Stat. Probabil. Lett., 4, p. 319–323.
29. Tweedie, M. C. K. (1957a). Statistical properties of Inverse Gaussian distribution I, Annals of Mathematics & Statistics, 28, p. 362-77.
30. Tweedie, M. C. (1957b). Statistical properties of inverse Gaussian distribution II, Annals of Mathematical Statistics, 28(3), p. 696-705.