Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666
Vol. 9, Issue 1 (2016):11-19

ECONOMIC DESIGN OF $\overline{X}$ CONTROL CHART UNDER MEASUREMENT ERROR

J. R. Singh and Manzoor A. Khanday*
School of Studies in Statistics, Vikram University, Ujjain, India
E Mail: Corresponding Author: *manzoorstat@gmail.com

Received September 01, 2015
Modified February 27, 2016
Accepted April 04, 2016

Abstract

In most of the industrial situations, data follow normal distribution. We may be confronted with an industrial situation where the assumption of normality and measurement errors are achievable or desirable. Thus, there is a need for a procedure which enables us to deal with measurement errors of the data. The design of a control chart requires the engineer or analyst to decide a sample size, a sampling interval and the control limit, to design control charts accordingly and to continue our search for the assignable causes of variation. The objective of this paper is to determine the design parameters, namely, sample size (n) and sampling interval (h) between successive samples. A numerical illustration has been supported to investigate the effects of cost parameters on the solution of the design. It may be inferred that measurement errors affect considerably the optimum value of the sample size and optimum sampling interval. It is necessary to point out that the measurement errors of the population should be taken into account while designing a control chart as the optimum values of the control chart parameters are affected by the measurement errors of the population. Visual comparisons of OC and ARL curves have been drawn in support of the problem.

Key Words: Economic Design Control Chart, Correlation, OC Function, Average Run Length, Measurement Error.

1. Introduction

Statistical Quality Control (SQC) techniques aim at improving the quality of manufactured products at a reasonable low cost. The two important tools of SQC are control charts and sampling plans. Measured quality of manufactured product is always subject to ascertain amount of variation as a result of chance. Some stable “system of chance causes” is inherent in any particular scheme of production and inspection. Variations within this stable pattern are inevitable. However, the reasons for variations outside this stable pattern may be discovered and corrected. In industry today, the form in which applied statistics is most widely used is that of control charts. A control chart is statistical device principally used to differentiate between the causes of variation in quality. Thus Statistical Quality Control refers to the use of statistically based methods to monitor, control, evaluate, analyze and improve process in production system, but errors occurs in any particular production process. Thus measurement error plays a vital role in control chart and the process variability is observed in any control chart or in any particular manufacturing process which is the mixture of inherent variability in the processes and the error due to the measurement tool. Kanazuka (1986) discusses if the measurement error is large comparative to the process variability, the control chart to detect any shift in the process level is affected. Ryan (2011) addresses a discussion on the measurement error and its effect on the performance on control charts. The effect of measurement errors for $\overline{X}$ chart was discussed by Bennett (1954), Abraham (1977), Mizuno (1961), Mittag and Stemann (1998). Rahim (1985) observed the effect of non-normality and measurement errors on the economic design of $\overline{X}$ charts. Huwang and Hung (2007) considered the effect of measurement error on the control charts for monitoring multivariate process variability. Khanday and Singh (2015) studied the effect of Markoff's model on Economic design of $\overline{X}$ control charts under independent observations. Walden (1990) measured the power of $\overline{X}$, R and $\overline{X}$ - R charts using ARL when measurement error affects the system. Stemann and Weihs (2001) and Maravelakis et al. (2004) investigated the effect of measurement error on the EWMA chart. Yang (2002) investigated the effect of measurement error on the asymmetric economic design and S control charts. Maravelakis (2012) considered the aged problem and investigated the effect of measurement error on the performance of the CUSUM control chart for the mean. More recently, Yang et al. (2013) proposed a new EWMA control chart to monitor the exponentially distributed service time between consecutive events with the measurement error instead of monitoring the number of events in a given time interval. The questioning of interaction between quality and manufacturing operation has been addressed recently by Gershwin and Kim (2005), and Colledani (2008). Their studies are the first investigations of how quality considerations can modify the production control. The design of control charts involves the selection of three parameters: sampling size (n), control frequency (h), and control limits (L) in order to detect earlier tools and processes shifts (Montgomery (2004)). Thus, Economic design of control charts is a method which aims at determining these parameters of a control chart in optimizing a cost function of the process monitored. A breakthrough has been the generalization of all these models by Lorenzen and Vance (1986), it is nowadays a reference in economic design, as it can be easily implemented and adapted. The purpose of this paper is to study the effect of measurement error on economic design of $\overline{X}$ control chart for independent data and to obtain the values of sample size n, sampling interval h, average run length (ARL) and OC values. Mathematical investigation has been used for calculating the design parameters and effects of measurement error on the economic design of $\overline{X}$ control charts have been observed.

2. Mathematical model for the cost function

Duncan (1956) obtained an approximate function for the average net income per hour of using the control chart for mean of normal variables as:

$$I = V_0 - \frac{\eta MB + (\alpha T / h) + \eta W}{1 + \eta B} - \frac{b + cn}{h} \tag{2.1}$$

Duncan's cost model indicates

(i) the cost of an out-of –control conditions,

(ii) the cost of false alarms,

(iii) the cost of finding an assignable cause and,

(iv) the cost of sampling inspection, evolution, and plotting.

Notations

$V_0 =$ the average per hour income when process is in control and process average is μ,

$V_1 =$ the average per hour income when process is not in control and process average is $\mu' = \mu + \delta\sigma$,

$M = V_0 - V_1$

$\eta =$ the average number of times the assignable cause occur within an interval of time,

$B = ah + Cn + D,$

$a = \frac{1}{P} - \frac{1}{2} + \frac{\eta h}{12},$

$h =$ Sampling interval in hours,

$Cn =$ the time required to take and inspect a sample of size n,

$D =$ average time taken to find the assignable cause after a point plotted on the chart falls outside the control limits,

$P =$ Probability of detecting an assignable cause when it exists,

$$= \int_{-\infty}^{\mu - \frac{k\sigma}{\sqrt{n}}} g(\overline{x}/\mu')d\,\overline{x} + \int_{\mu + \frac{k\sigma}{\sqrt{n}}}^{\infty} g(\overline{x}/\mu')d\,\overline{x}$$

$$\cong 1 - \Phi(k - \delta\sqrt{n}) \quad \text{for } \delta > 0$$

Where $g(\overline{x}/\mu')$ is the density function of $\overline{x}$ when the true mean μ and $\Phi(x)$ is the normal probability

$\alpha =$ probability of wrongly indicating the presence of assignable cause.

$$= \int_{\mu - \frac{k\sigma}{\sqrt{n}}}^{\mu + \frac{k\sigma}{\sqrt{n}}} g(\overline{x}/\mu)d\,\overline{x} = 2\Phi(-k) \tag{2.2}$$

$T =$ The cost per occasions of looking for an assignable cause when no assignable cause exists,

$W =$ the average cost per occasion of finding the assignable cause when it exist,

$b =$ per sample cost of sampling and plotting, that is independent of sample size,

and c = the cost per unit of measuring an item in a sample.

The average cost per hour involved for maintaining the control chart is $\frac{(b + cn)}{h}$. The average net income per hour of the process under the surveillance of the control chart for mean can be rewritten as,

$$I = V_0 - L$$

Where

$$L = \frac{\eta MB + (\alpha T / h) + \eta W}{1 + \eta B} + \frac{b + cn}{h} \tag{2.3}$$

L Can now be treated as the per hour cost due to the surveillance of the process under the control chart. The probability density function is determined from the sampling distribution of mean and are written as:

$$P = 1 - \Phi(\theta) \tag{2.4}$$

Where, $\theta = (k - \delta\sqrt{n})$ (2.5)

3. Derivation for optimum value of sample size n and sampling interval h

One can determine the optimum values of sample siz n and sampling interval h either by maximizing the gain function I or by minimizing the cost function L with respect to n and h and we get,

$$\frac{\partial L}{\partial n}=\frac{(1+\eta B)\left(\eta M\frac{\partial B}{\partial n}+\frac{T}{h}\frac{\partial \alpha}{\partial n}\right)-\left(\eta MB+\frac{\alpha T}{h}+\eta W\right)\eta\frac{\partial B}{\partial n}}{(1+\eta B)^2}+\frac{c}{h}=0 \tag{3.1}$$

$$\frac{\partial L}{\partial h}=\frac{(1+\eta B)\left(\eta M\frac{\partial B}{\partial h}-\frac{\alpha T}{h^2}\right)-\left(\eta MB+\frac{\alpha T}{h}+\eta W\right)\eta\frac{\partial B}{\partial n}}{(1+\eta B)^2}-\left(\frac{b+cn}{h^2}\right)=0 \tag{3.2}$$

Where,

$$\frac{\partial B}{\partial n}=\frac{-h}{P^2}\frac{\partial P}{\partial n}+c \tag{3.3}$$

$$\frac{\partial B}{\partial h}=\frac{1}{P}-\frac{1}{2}+\frac{\eta h}{6} \tag{3.4}$$

$$\frac{\partial \alpha}{\partial n}=0 \tag{3.5}$$

$$\frac{\partial P}{\partial n}=\frac{\delta}{2\sqrt{n}}\phi(\theta) \tag{3.6}$$

The solutions of the equations (3.1) and (3.2) for n and h are:

$$\eta h\left(M-\eta MB-\frac{\alpha T}{h}-\eta W\right)\frac{\partial B}{\partial n}+\frac{T}{h}\frac{\partial \alpha}{\partial n}+\eta B\left(\eta M\frac{\partial B}{\partial n}+\frac{T}{h}\frac{\partial \alpha}{\partial n}\right)+c(1+\eta B)^2=0 \tag{3.7}$$

$$\eta h^2\left(M-\eta MB-\frac{\alpha T}{h}-\eta W\right)\frac{\partial B}{\partial h}-\alpha T(1+\eta B)+\eta^2h^2MB\frac{\partial B}{\partial n}-(b+cn)(1+\eta B)^2=0 \tag{3.8}$$

By assuming η to be small and noting that the optimum h is roughly of order of $\frac{1}{\sqrt{\eta}}$, we neglect terms with ηB containing ηWc, $\frac{\alpha T}{h}$ and the terms equating higher powers of η. The equations (3.7) and (3.8) are simplified and put in the following form:

$$\frac{-\eta h^2 M}{P^2}\frac{\partial P}{\partial n}-\eta\alpha T+c=0 \tag{3.9}$$

$$\eta Mh^2\left(\frac{1}{P}-\frac{1}{2}\right)-(\alpha T+b+cn)=0 \tag{3.10}$$

From the equation (3.10) we get,

$$h=\left(\frac{\alpha T+b+cn}{\eta M\left(\frac{1}{P}-\frac{1}{2}\right)}\right)^{\frac{1}{2}} \tag{3.11}$$

By eliminating h from the equation (3.9), we get,

$$-\frac{\alpha T+b+cn}{P^2\left(\frac{1}{P'}-\frac{1}{2}\right)}.\frac{\partial P}{\partial n}-\eta\alpha T+c=0 \qquad (3.12)$$

The values of n for which the equation (2.12) satisfy yield us the required optimum value of sample size n. Substituting this value of n equation (2.11), we find the optimum value of the sampling interval h.

4. Description for optimum value of sample size n and sampling interval h under measurement error

Assuming that the true measurement x and the random error of measurement e are additive, then

$$X=x+e \qquad (4.1)$$

The mean and standard deviation of the observed measurement X can be written as:

$E(X)=\mu$, where μ is the mean of x and e ~N (0, σ_e^2),

V(X) = V(x) +V (e) = $\sigma_p^2+\sigma_e^2=\sigma_x^2$ (say)

The correlation coefficient ρ between the true and observed measurement is given by:

$$\rho=\frac{E\left[(x-\mu)(X-\mu)\right]}{\sigma_p\sigma_x}$$

$$\rho=\frac{r}{\sqrt{1+r^2}}= \text{, where } r=\frac{\sigma_p}{\sigma_e} \qquad (4.2)$$

Therefore the probability density function under measurement error for independent case is:

$$P_e=1-\Phi(\theta_e) \qquad (4.3)$$

Where $\theta_e=\rho(k-\delta\sqrt{n})$ (4.4)

$$\alpha'=2\Phi(-\rho k)$$

In presence of measurement error the equation (3.1) and (3.2) will reduce in following form

$$\frac{\partial L}{\partial n}=\frac{(1+\eta B)\left(\eta M\frac{\partial B}{\partial n}+\frac{T}{h}\frac{\partial\alpha_e'}{\partial n}\right)-\left(\eta MB+\frac{\alpha_e' T}{h}+\eta W\right)\eta\frac{\partial B}{\partial n}}{(1+\eta B)^2}+\frac{c}{h}=0 \qquad (4.5)$$

$$\frac{\partial L}{\partial h}=\frac{(1+\eta B)\left(\eta M\frac{\partial B}{\partial h}-\frac{\alpha_e' T}{h^2}\right)-\left(\eta MB+\frac{\alpha_e' T}{h}+\eta W\right)\eta\frac{\partial B}{\partial n}}{(1+\eta B)^2}-\left(\frac{b+cn}{h^2}\right)=0 \qquad (4.6)$$

where,

$$\frac{\partial B}{\partial n}=\frac{-h}{P_e^2}\frac{\partial P_e}{\partial n}+c$$

$$\frac{\partial B}{\partial h}=\frac{1}{P_e}-\frac{1}{2}+\frac{\eta h}{6}$$

$$\frac{\partial \alpha_e^{'}}{\partial n} = 0$$

$$\frac{\partial P_e}{\partial n} = \frac{\rho\delta}{2\sqrt{n}}\phi(\theta_e) \tag{4.7}$$

Similarly by using the above procedure we get,

$$h = \left(\frac{\alpha_e^{'} T + b + cn}{\eta M\left(\frac{1}{P_e^{'}} - \frac{1}{2}\right)} \right)^{\frac{1}{2}} \tag{4.8}$$

and, $$-\frac{\alpha' T + b + cn}{P^2\left(\frac{1}{P'} - \frac{1}{2}\right)} \cdot \frac{\partial P}{\partial n} - \eta\alpha' T + c = 0 \tag{4.9}$$

The values of n for which the equation (4.9) satisfy yield us the required optimum value of sample size *n*. Substituting this value of n in equation (4.8), we find the optimum value of the sampling interval *h*.

δ	k=3		k=2.5		k=2		k=1.5		k=1	
	n	h	n	h	n	h	n	h	n	h
0.5	88	4.14255	72	3.9012	66	4.251	131	6.360737	106	7.33813
1	23	2.33708	20	2.413	19	3.026	35	4.615166	29	6.20561
1.5	11	1.80259	9	1.9958	9	2.706	16	4.19999	14	5.95488
2	6	1.56478	6	1.8152	6	2.574	9	4.031122	8	5.859

Table 1: Values of optimal sample size n and sampling interval h

δ	k=3	k=2.5	k=2	k=1.5	k=1
0.5	0.95388	0.9581	0.981	0.99999	0.99998
1	0.96382	0.9727	0.991	0.99999	0.99999
1.5	0.97309	0.9824	0.995	1	1
2	0.98009	0.9881	0.997	1	1

Table 2: OC values under different values of k and δ

ARL	k=3	k=2.5	k=2	k=1.5	k=1
$1/\alpha$	370.37	80.6452	21.978	7.485	3.15159

Table 3: ARL Values for different values of k

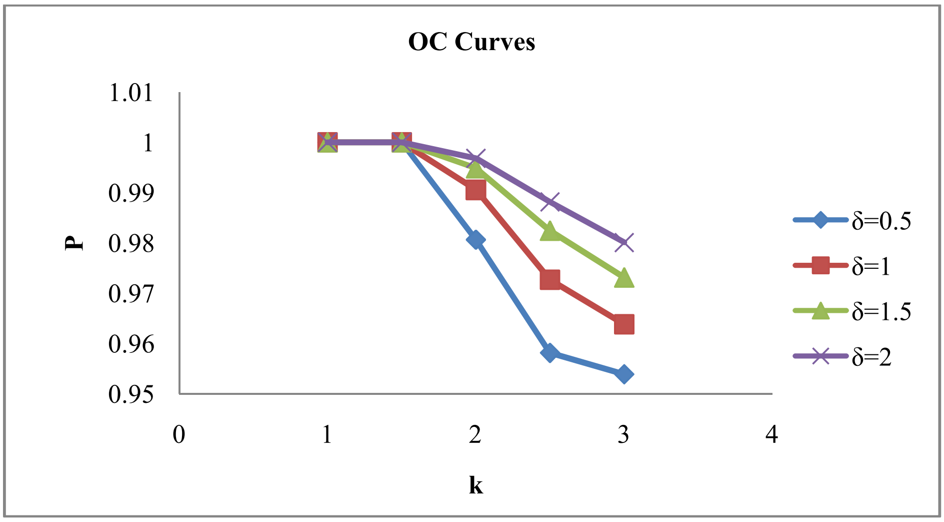

Fig. 1: OC curves at different values of δ

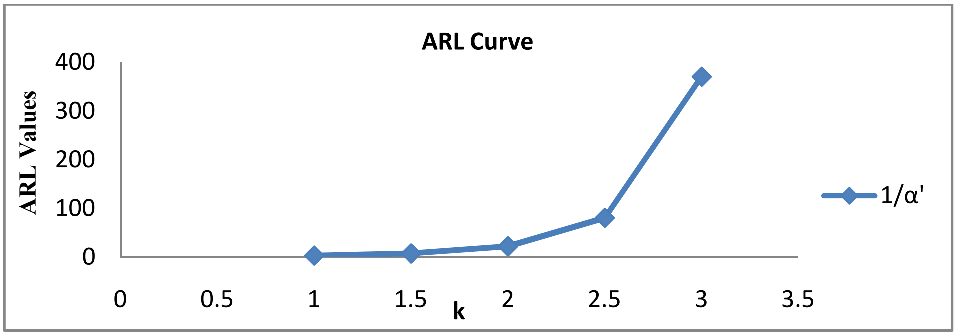

Fig. 2: ARL Curve for different values of k.

5. Numerical Illustration and Conclusion

For the purpose of numerical illustration, we take k=3, 2.5, 2, 1.5 and 1, δ =0.5, 1.0, 1.5, and 2.0, η =0.01, M=100, W=25, T=50, c=0.05, D=2, b=0.5 and r=0 (for independence), we get $\rho = 0$ from equation (4.2) and determine the optimum values of sample size n and sampling interval h which are presented in Table 1. The sample size required to detect given shift increases with the increase in the value of k although the sampling interval is not much affected. This is more marked for detecting small shifts in the process average. On the other hand while the values of δ are decreasing the OC curves becomes steeper which is clearly seen in visual comparison of the OC curves in Fig. 1. Thus we conclude that chart remains effective when we choose smaller values of δ also the average run length increases with the increase of k. From the visual comparison it is seen that after every 370 samples there is an indication false alarms when the value of k=3, and it should be noted that under data dependence the value of k must be less or equal to three. If we increases the values of k (> 3) the defective lots will come which is bad for the consumer, as we are confronted with an industrial situation where the assumption of error free measurements are achievable or desirable. Thus, there is a need for a procedure which enables us to deal with measurement errors of the data, to design control charts accordingly and to continue our search for the assignable causes of variation. It may be inferred that measurement errors affect considerably the optimum value of the sample size and optimum sampling interval. It is necessary to point out that the measurement errors of the population should be taken into account while designing a control chart as the optimum values of the control chart parameters are affected by the measurement errors of the population.

Acknowledgment

Authors are greatly indebted to the reviewers and the Editorial Board of the journal for their suggestions in improving the quality of this paper.

References

1. Abraham, B. (1977). Control charts and measurement error. Milwaukee, WI: ASQC Technical Conference Transactions, American Society for Quality Control, p. 370-374.
2. Bennet, C. A. (1954). Effect of measurement error on chemical process control, Industrial Quality Control, 10, p. 17-20.
3. Colledani, M. (2008). Integrated analysis of quality and production logistics performance in manufacturing lines, In Proceedings of IFAC world congress, 6–11 July Seoul, Korea, 8368, p. 1–7.
4. Duncan, A. J. (1956). The economic design of $\overline{X}$ chart used to maintain current control of a process, Journal of the American Statistical Association, 51, p. 228-242.
5. Gershwin, S. and Kim, J. (2005). Integrated quality and quantity modelling of a production line, OR Spectrum, 27 (2-3), p. 287–314.
6. Huwang, L., & Hung, Y. (2007). Effect of measurement error on monitoring multivariate process variability, Statistica Sinica, 17, p. 749-760.

7. Kanazuka, T. (1986). The effects of measurement error on the power of $\overline{X}$ - R charts, Journal of Quality Technology, 18, p. 91-95.
8. Khanday, M. A. and Singh, J. R. (2015). Effect of Markoff's model on Economic design of $\overline{X}$ control chart for independent observations. International Journal of Scientific Research in Mathematical and Statistical Sciences, 2(2), p.1-6.
9. Lorenzen, T. J. Vance, L. C. (1986). The economic design of control charts: a unified approach, Technometrics, 28(1), p. 3–10.
10. Maravelakis, P. E. (2012). Measurement error effect on the CUSUM control chart, Journal of Applied Statistics, 39, p. 323-336.
11. Mittag, H. J. and Stemann, D. (1998). Gauge inspection effect on the performance of the $\overline{X}$ - S control chart, Journal of Applied Statistics, 25, p. 307-317.
12. Mizuno, S. (1961). Problems on measurement errors in process control, Bulletin of the International Statistical Institute, 38, p. 405-415.
13. Montgomery, D. (2004). Introduction to statistical quality control. New York: John Wiley & Sons.
14. Rahim, M. A. (1985). Economic model of charts under non-normality and measurement error, Computers and Operations Research, 12, p. 291-299.
15. Ryan, T. P. (2011). Statistical Methods for Quality Improvement, Third Edition. New York: John Wiley and Sons.
16. Stemann, D. and Weihs, C. (2001). The EWMA-X-S control chart and its performance in the case of precise and imprecise data, Statistical Papers, 42, p. 207-223.
17. Walden C. T. (1990). An analysis of variable control charts in the presence of measurement error, Unpublished Master's Thesis, Department of Industrial Engineering, Mississippi State University, USA.
18. Yang, S-F. (2002). The effects of imprecise measurement on the economic asymmetric and S control charts, The Asian Journal on Quality, 3, p. 46-55.
19. Yang, S-F., Yang, C-M. and Cheng, G-F. (2013). Effects of measurement error on process signal detection, Applied Mechanics and Materials, p. 263-266, p. 496-500.