Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666
Vol. 9, Issue 1 (2016): 47-58

PARAMETRIC INFORMATION GENERATING FUNCTION WITH UTILITIES

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Received September 07, 2015
Modified March 15, 2016
Accepted April 15, 2016

Abstract

In this paper, we have defined parametric relative information generating function with utilities. We have also discussed its particular and limiting cases. It is interesting to note that differentiation of this relative information generating function at t=0 produces various well known measures of information. The relative information generating function for different distributions have also been studied.

Key Words: Information Generating Function, Discrete Probability Distribution, Utility Distribution.

1. Introduction

The concept of information generating function of a probability distribution was given by Golomb [2] for Shannon's [10] measure of entropy and Kullback-Leiblers's [7] measure of relative information. Golomb [2] defined the information generating function as

$$I(t) = -\sum_{i\in N} p_i^t \quad , t \geq 1 \tag{1.1}$$

where $\{p_i\}$ is a complete probability distribution with $i \in N$,N is a discrete sample space and t is a real or complex variable. Further it may be noted that

$$\frac{\partial I(t)}{\partial t}|_{t=1} = H(P) = -\sum_{i\in N} p_i \log p_i \tag{1.2}$$

where H (P) is a Shannon's entropy [10], $\{p_i\}$ are probability attached to the events $\{E_i\}$.The quantity (1.2) measures average information but does not take into account the relative information of the events. Belis and Guiasu [1] introduced the measure of useful information

$$H(P;U) = -\sum_{i\in N} U_i p_i \log p_i \tag{1.3}$$

where $\{u_i\}$ is the utility distribution and $u_i>0$ is the utility attached to the i^{th} event which occur with probability p_i.

Hooda and Bhakar [5] gave mean value characterization of the following 'useful information measures:

$$H(P;U) = {-\sum_{i\in N} U_i p_i \log p_i} \Big/ {\sum_{i\in N} U_i p_i} \tag{1.4a}$$

and $H_\alpha(P;U) = \frac{1}{\alpha-1}\log \sum_{i\in N} U_i p_i^\alpha / \sum_{i\in N} U_i p_i$ (1.4b)

New mean value characterization of the following 'useful' information measures:

$$H_\alpha(P;U) = \frac{1}{\alpha-1}\log \left[\sum_{i\in N} U_i p_i^\alpha / \sum_{i\in N} U_i p_i\right] \quad (1.4c)$$

Mahajan and Kumar (2015) have also defined useful information generating function:

$$I(P;U,t) = -\sum_{i\in N}(U_i\, p_i)^t \Big/ \sum_{i\in N} U_i p_i \quad (1.5)$$

where $P = \{p_1, p_2, \ldots.., p_n\}$ and $U = \{u_1, u_2, \ldots.., u_n\}$ are respectively probability and utility distributions and t is a real or complex variable. They have studied the properties of (1.5) in next section and derived the information generating function for some probability distributions in section 3.

2. Parametric information generating function

Mahajan and Kumar (2015) have defined a new parametric information generating function information generating function:

$$I(P;U,t) = -\sum_{i\in N}(U_i\, p_i)^t \Big/ \sum_{i\in N} U_i p_i$$

Since $0 \le p_i \le 1 \ \forall i$ and < u_i > is bounded for an experiment, more over being positive term series (1.4a) is absolutely convergent $\forall t \ge 1$ and also it converges uniformly and therefore each term of the series possess continuous derivative .So, the derivative of (1.4a) at t=1 w.r.t. t is:

$$I(P;U,t) = -\sum_{i\in N}(U_i\, p_i)^t \Big/ \sum_{i\in N} U_i p_i \quad (2.1)$$

which is (1.4a) and has found wide applications in Economics, Accountancy etc.

In case the utilities are ignored or u_i=1 for each i, (1.4a) reduces to (1.2) because $\sum p_i = 1$.

Suppose $P = \left\{(p_1, p_2, \ldots.., p_n), 0 \le p_i \le 1, \sum_{i=1}^{n} p_i = 1\right\}$,be a discrete probability distribution of a set of events E= {E_1, E_2,……. E_n} of a discrete infinite sample space N on the basis of an experiment having utility distribution U={$(u_1, u_2, u_3 \ldots\ldots\ldots u_n)$; $u_i > 0 \ \ \forall$ i }, where N is discrete sample space.

We know that weighted mean of u_i and p_i is given by:

$$\frac{\sum_{i=1}^{n} U_i p_i}{\sum_{i=1}^{n} U_i} \quad (2.2)$$

If we replace U_i with weights $(U_i p_i)^{\beta_i}$ and p_i of order $\alpha - 1$ then we get new weighted mean of order $\alpha - 1$ as:

$$M_{\alpha,\beta}(P;U)=\left[\sum_{i=1}^{n}(u_ip_i)^{\beta_i}p_i^{\alpha-1}\Big/\sum_{i=1}^{n}(U_ip_i)^{\beta_i}\right]^{1/\alpha-1}$$

$$M_{\alpha,\beta}(P;U)=\left[\sum_{i=1}^{n}U_i^{\beta_i}p_i^{\alpha+\beta_i-1}\Big/\sum_{i=1}^{n}(U_ip_i)^{\beta_i}\right]^{1/\alpha-1}\quad;\alpha\geq 0,\alpha\neq 1,\beta_i\geq 1$$

(2.3)

For which we have the generalized useful information generating function given by

$I_{\alpha,\beta}(P;U,t)=\left[M_{\alpha,\beta_i}(P;U)\right]^t$

From equation (2.3) we get

$$I_{\alpha,\beta}(P;U,t)=\left[\sum_{i=1}^{n}U_i^{\beta_i}p_i^{\alpha+\beta_i-1}\Big/\sum_{i=1}^{n}(U_ip_i)^{\beta_i}\right]^{t/\alpha-1}$$

(2.4)

where *t* is a real or complex variable.

On differentiating equation (2.4) w.r.t. *t* at *t=0* respectively, we have:

$$H_\alpha^{\beta_i}(P,U)=(\alpha-1)^{-1}\log\left[\sum_{i=1}^{n}U_i^{\beta_i}p_i^{\alpha+\beta_i-1}\Big/\sum_{i=1}^{n}(U_ip_i)^{\beta_i}\right]$$

$$H_\alpha^{\beta_i}(P,U)=\frac{1}{\alpha-1}\log\left[\sum_{i=1}^{n}U_i^{\beta_i}p_i^{\alpha+\beta_i-1}\Big/\sum_{i=1}^{n}(U_ip_i)^{\beta_i}\right]\tag{2.5}$$

which is the generalized useful information measure of order α and type β_i.

When $\beta_i=\beta$ for each *i* then equation (2.5) reduces to

$$H_\alpha^{\beta}(P,U)=\frac{1}{\alpha-1}\log\left[\sum_{i=1}^{n}U_i^{\beta}p_i^{\alpha+\beta-1}\Big/\sum_{i=1}^{n}(U_ip_i)^{\beta}\right]\tag{2.6}$$

which is the generalized useful information measure of order α and type β.

If $\beta=1$ then equation (2.6) reduces to (1.4c)

$$H_\alpha^{\beta}=\frac{1}{\alpha-1}\log\left[\sum_{i=1}^{n}U_i\,p_i^{\alpha}\Big/\sum_{i=1}^{n}(U_ip_i)\right]\tag{2.6a}$$

3. Useful information generating functions and information measures

In this section, we have derived useful information generating functions and corresponding useful relative information measures for uniform, geometric, Poisson and exponential distributions as particular examples.

Example 3.1: For the Uniform probability distribution $(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$ after experiment, predicted probability distribution $(\frac{1}{M}, \frac{1}{M}, \ldots, \frac{1}{M})$ before experiment and the utility distribution (u,u,u,…,u) of an experiment, the equation (2.6) reduces as:

$$I_{\alpha,\beta}(P; U, t) = \left[\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha+\beta_i-1} \Big/ \sum_{i=1}^{n} (U_i p_i)^{\beta_i}\right]^{t/\alpha-1}$$

By putting $\beta_i = \beta = 1$, $U_i = U$ and $p_i = \frac{1}{N}$, then we have,

$$I_{\alpha,\beta}(P; U, t) = \left[\sum_{i=1}^{n} U\left(\frac{1}{N}\right)^{\alpha} \Big/ \sum_{i=1}^{n} \left(U\frac{1}{N}\right)^{1}\right]^{t/\alpha-1}$$

$$I_{\alpha,\beta}(P; U, t) = \left(\frac{1}{N}\right)^{t}$$

Also equation (2.6a) reduces to

$$\frac{\partial}{\partial t} I_{\alpha,\beta}(P; U, t) = H_{\alpha}^{\beta}(P, U) = \frac{1}{\alpha-1}\log\left[\sum_{i=1}^{n} U_i p_i^{\alpha} \Big/ \sum_{i=1}^{n} (U_i p_i)\right]$$

$$= \frac{1}{\alpha-1}\log\left[\sum_{i=1}^{n} U\left(\frac{1}{N}\right)^{\alpha} \Big/ \sum_{i=1}^{n} U\left(\frac{1}{N}\right)\right]$$

$$= \frac{\alpha-1}{\alpha-1}\log\left(\frac{1}{N}\right)$$

$$\frac{\partial}{\partial t} I_{\alpha,\beta}(P; U, t) = H_{\alpha}^{\beta}(P, U) = -\log N$$

Example 3.2: Geometric distribution: Consider the Geometric distribution $(q, qp, qp^2, \ldots)$, $p+q=1$ and Geometric utility distribution $(v, vu, vu^2, \ldots)$. It is the most general case when utility also follows geometric distribution. Then we have,

$$I\alpha,\beta(P;U;t) = \left[\sum_{i=1}^{n} v(1, u, u^2, \ldots)^{\beta_i} q(1, p, p^2, \ldots)^{\alpha+\beta_i-1} \Big/ \sum_{i=1}^{n} qv(1, u, u^2, \ldots)(1, p, p^2, \ldots)\right]^{t/\alpha-1}$$

Put $\beta_i = \beta = 1$

$$= \left[vq\sum_{i=1}^{n}(1, u, u^2, \ldots)(1, p, p^2, \ldots)^{\alpha} \Big/ vq\sum_{i=1}^{n}(1, u, u^2, \ldots)(1, p, p^2, \ldots)\right]^{t/\alpha-1}$$

$$= \left[\left(\frac{1}{1-u}\right)\left(\frac{1}{1-p}\right)^{\alpha} \Big/ \left(\frac{1}{1-u}\right)\left(\frac{1}{1-p}\right) \right]^{t/\alpha-1}$$

$$= I\alpha,\beta(P;U;t) = \left(\frac{1}{q}\right)^{t}$$

Also equation (2.6a) reduces to

$$\frac{\partial}{\partial t} I_{\alpha,\beta}(P;U,t) = H_{\alpha}^{\beta}(P,U) = \frac{1}{\alpha-1}\log\left[\sum_{i=1}^{n} U_i p_i^{\alpha} \Big/ \sum_{i=1}^{n}(U_i p_i) \right]$$

$$= \frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n} V(1,U,U^2,U^3\ldots)q(1,p,p^2,p^3\ldots)^{\alpha}}{\sum_{i=1}^{n} V(1,U,U^2,U^3\ldots)q(1,p,p^2,p^3\ldots)}\right]$$

$$= \frac{1}{\alpha-1}\log\left[\frac{V\left(\frac{1}{1-U}\right)q\left(\frac{1}{1-p}\right)^{\alpha}}{V\left(\frac{1}{1-U}\right)q\left(\frac{1}{1-p}\right)}\right]$$

$$\Rightarrow \frac{\partial}{\partial t} I_{\alpha,\beta}(P;U,t) = -\log q$$

Example 3.3: For the exponential distribution with mean 1/λ and exponential utility distribution with mean 1/μ, we consider,

$p(x) = \lambda e^{-\lambda x}$, λ > 0 , 0≤x< ∞ and

$u(x) = \mu e^{-\mu y}$, μ> 0 , 0≤y< ∞

From equation (2.4) after putting β = 1, we have

$$I_{\alpha,\beta}(P;U,t) = \left[\frac{\sum_{i=1}^{n} \mu e^{-\mu y}\left(\lambda e^{-\lambda x}\right)^{\alpha}}{\sum_{i=1}^{n}\{(\mu e^{-\lambda y})(\lambda e^{-\lambda x})\}}\right]^{\frac{t}{\alpha-1}}$$

$$= \left[\left(\lambda e^{-\lambda x}\right)^{\alpha-1}\right]^{\frac{t}{\alpha-1}}$$

Also from equation (2.6a)

$$\frac{d}{dt} I_{\alpha,\beta}(P;U,t) = \frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n} U_i p_i^{\alpha}}{\sum_{i=1}^{n}(U_i p_i)}\right]$$

$$= \frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n} \mu e^{-\mu y}\left(\lambda e^{-\lambda x}\right)^{\alpha}}{\sum_{i=1}^{n}(\mu e^{-\mu y}\lambda e^{-\lambda x})}\right]$$

$$= \frac{\alpha-1}{\alpha-1}\log \lambda e^{-\lambda x}$$

$$\frac{d}{dt} I_{\alpha,\beta}(P;U,t) = \log\lambda - \lambda x$$

Example 3.4: For the Poisson distribution, we consider,

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; \quad x = 0,1,2,\ldots ; \lambda > 0$$

$$u(x) = \frac{e^{-\mu}\mu^{y}}{y!} \quad ; \quad y = 0,1,2,\ldots ; \mu > 0$$

After substitution these values and β =1 in equation (2.4), we get

$$I_{\alpha,\beta}(P;U,t) = \left[\frac{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)^{\alpha}}{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)}\right]^{\frac{t}{\alpha-1}}$$

$$= \left[\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)^{\alpha-1}\right]^{\frac{t}{\alpha-1}}$$

$$I_{\alpha,\beta}(P; U, t) = \left[\frac{e^{-\lambda}\lambda^{x}}{x!}\right]^{t}$$

From equation (2.6a), we get

$$\frac{d}{dt} I_{\alpha,\beta}(P; U, t) = \frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)^{\alpha}}{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)}\right]$$

$$= \frac{1}{\alpha-1}\log\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)^{\alpha-1}$$

$$\frac{d}{dt} I_{\alpha,\beta}(P; U, t) = \log\frac{e^{-\lambda}\lambda^{x}}{x!}$$

4. Two Generalized 'Useful' relative information functions

Suppose $P= \{ (p_1,p_2,\ldots p_n),\ 0 < p_i \le 1, \sum_i^n p_i=1\}$ be a discrete probability distribution whose predicted probability distribution is

$Q = \{(q_1,q_2,\ldots q_n),\ 0< q_i\le 1, \sum_i^n q_i=1\}$

And $U= \{(u_1,u_2,\ldots u_n),\ 0< u_i\ , \forall\ i\}$ is the utility distribution of a discrete sample space N.

Let us consider,

$I(P,Q,t) = \sum_i^n (\frac{p_i}{q_i})^{p_i u_i (t-1)}$

$\frac{dI}{dt}/t=1 = \sum_i^n p_i u_i \log (\frac{p_i}{q_i})$

When $u_i = 1$ then

$$H(P/Q;U) = \sum_i^n p_i \log (\frac{p_i}{q_i}) \tag{4.1}$$

which is the same expression of Kullback-Leibler relative entropy.

We know that U_i, p_i and q_i is given by:

$$\frac{\sum_{i=1}^{n} U_i p_i q_i}{\sum_{i=1}^{n} U_i} \tag{4.2}$$

If we replace U_i with weights $(U_i p_i)^{\beta_i}$ and p_i and q_i of order $(\alpha-1)$ then we get new weighted mean of order $\alpha-1$ is:

$$M_{\alpha,\beta}(P/Q\ , U)=\left[\sum_{i=1}^{n}(u_i p_i)^{\beta_i}\, p_i^{\alpha-1} q_i^{\alpha-1} \Big/ \sum_{i=1}^{n}(U_i p_i)^{\beta_i}\right]^{1/\alpha-1}$$

$$M_{\alpha,\beta}(P;\ U)=\left[\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha-1+\beta_i}\, q_i^{\alpha-1} \Big/ \sum_{i=1}^{n} U_i^{\beta_i} p_i^{\beta_i}\right]^{1/\alpha-1} ;\alpha \ge 0 \quad ,\alpha \ne 1, \beta_i \ge 1 \tag{4.3}$$

for which we have the generalised useful information generating function given by:

$$I_{\alpha,\beta}(P/Q\ ; U\ ,t)=\left[M_{\alpha,\beta_i}(\frac{P}{Q};\ U)\right]^{t}$$

From equation (4.3), we get,

$$I_{\alpha,\beta}(P/Q;U,t)=\left[\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha-1+\beta_i} q_i^{\alpha-1} \Big/ \sum_{i=1}^{n} U_i^{\beta_i} p_i^{\beta_i}\right]^{t/\alpha-1} \tag{4.4}$$

where t is a real or complex variable.

On differentiating equation (2.4) w.r.t t at $t=0$ respectively. We have:

$$H_\alpha^{\beta_i}(P/Q;U) = (\alpha-1)^{-1}\log\left[\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha-1+\beta_i} q_i^{\alpha-1} \Big/ \sum_{i=1}^{n} U_i^{\beta_i} p_i^{\beta_i}\right] \tag{4.5}$$

Which is the new generalized useful relative information measure of order α and type β_i.

When $\beta_i = \beta$ for each i, (4.4) & (4.5) respectively reduce to the following :

$$I_{\alpha,\beta}(P/Q;t)=\left[M_{\alpha\beta}(P/Q;t)\right]^t=\left[\frac{\sum_{i=1}^{n} U_i^{\beta} p_i^{\alpha+\beta-1} q_i^{\alpha-1}}{\sum_{i=1}^{n} U_i p_i^{\beta}}\right]^{\frac{t}{\alpha-1}} \tag{4.6}$$

and $$H_\alpha^{\beta}(P/Q;U)=(\alpha-1)^{-1}\log\left[\frac{\sum_{i=1}^{n} U_i^{\beta} p_i^{\alpha+\beta-1} q_i^{\alpha-1}}{\sum_{i=1}^{n} (U_i p_i)^{\beta}}\right] \tag{4.7}$$

which is new generalized relative information measure of order α and β.

Particular cases :

i) If utilities are ignored or U_i for each i, equations (4.6) and (4.7) become

$$I_{\alpha\beta}(P/Q;t) = \left[\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{\alpha-1}}{\sum_{i=1}^{n} p_i^{\beta}}\right]^{\frac{t}{\alpha-1}} \tag{4.8}$$

$$I_{\alpha,\beta}(P/Q;t) = (\alpha-1)^{-1}\log\left[\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{\alpha-1}}{\sum_{i=1}^{n} p_i^{\beta}}\right] \tag{4.9}$$

which is the generalized measure of relative information.

ii) If we set $\beta = 1$ in (4.8), we have:

$$I_\alpha(P/Q;t) = \left[\frac{\sum_{i=1}^{n} p_i^{\alpha} q_i^{\alpha-1}}{\sum_{i=1}^{n} p_i}\right]^{\frac{t}{\alpha-1}}, \quad \text{since } \sum_{i=1}^{n} p_i = 1$$

$$I_\alpha(P/Q;U) = \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{\alpha-1}\right]^{\frac{t}{\alpha-1}}$$

$$I_\alpha(P/Q;U) = \left[\sum_{i=1}^{n} (p_i^{\alpha})^{\frac{t}{\alpha-1}} q_i^{t}\right] \tag{4.10}$$

which is new generalized relative information measure.

On differentiating (4.10) at $t = 1$ and $\alpha \to 1$, we get,

$$H(P/Q) = \sum_{i=1}^{n} p_i q_i \log p_i q_i \tag{4.11}$$

which is the Shannon Entropy for two generalized relative function.

iii) In case $\beta = 1$ in equation (4.4),we have,

$$I_\alpha(P/Q;U,t) = [M_\alpha(P/Q;U)]^t$$

$$I_\alpha(P/Q;U,t) = \left[\frac{\sum_{i=1}^{n} U_i p_i^{\alpha} q_i^{\alpha-1}}{(\sum_{i=1}^{n} U_i p_i)}\right]^{\frac{t}{\alpha-1}} \tag{4.12}$$

which is generalized 'useful' relative information generating function of order α and differentiation at t=0, we get,

$$H_\alpha(P/Q;U) = \left[\frac{\sum_{i=1}^{n} U_i p_i^{\alpha} q_i^{\alpha-1}}{(\sum_{i=1}^{n} U_i p_i)}\right] \tag{4.13}$$

which is new generalized useful relative information measure of order α.

If we have $U_i = 1$ and $\alpha \to 1$ then equation (4.13) reduces to (2.6a). Since $\sum_{i=1}^{n} p_i = 1$. Following Hardy, Littlewood and Polya[3] we can also have another Weighted mean of order $\alpha - 1$ of p_i, q_i and U_i
as given below :

$$M_{\alpha\beta}(P/Q,U) = \left[\frac{\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha\beta_i} q_i^{\alpha-1}}{\left(\sum_{i=1}^{n} U_i p_i\right)^{\beta_i}}\right]^{\frac{1}{\alpha-1}} ; \alpha \neq 1, \beta \geq 1, \alpha > 0 \qquad (4.14)$$

for which we have generalized useful relative information generating function given below

$$I_{\alpha,\beta}(P/Q,U,t) = [M(P/Q\,;U)]^t$$

$$= \left[\frac{\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha\beta_i} q_i^{\alpha-1}}{\left(\sum_{i=1}^{n} U_i p_i\right)^{\beta_i}}\right]^{\frac{t}{\alpha-1}} \qquad (4.15)$$

where t is a real or complex variable.

Differentiating equation (4.15) w.r.t. t at $t = 0$, we have :

$$I_{\alpha\beta_i}(P/Q\,;U) = (\alpha-1)^{-1} \log \left[\frac{\sum_{i=1}^{n} U_i^{\beta_i} p_i^{\alpha\beta_i} q_i^{\alpha-1}}{\left(\sum_{i=1}^{n} U_i p_i\right)^{\beta_i}}\right]$$

$$= H_{\alpha}^{\beta_i}(P/Q\,;U) \qquad (4.16)$$

which is a new measure and is called as the generalized useful relative measure of order α and β_i. When $\beta_i = \beta$ for
each i equation (4.15) and (4.16) reduces to

$$I_{\alpha,\beta}(P/Q;U,t) = \left[\frac{\sum_{i=1}^{n} U_i^{\beta} p_i^{\alpha\beta} q_i^{\alpha-1}}{\left(\sum_{i=1}^{n} U_i p_i\right)^{\beta}}\right]^{\frac{t}{\alpha-1}} \qquad (4.17)$$

which is also a new measure and can be called the generalized measure of "useful " relative information of order α and β.

Particular cases :

i) If utilities are ignored or $U_i = 1$ for each i in (4.15), we have,

$$I_{\alpha,\beta}(P/Q,t) = \left[\frac{\sum_{i=1}^{n} (p_i)^{\alpha\beta} (q_i)^{\alpha-1}}{\sum_{i=1}^{n} (p_i)^{\beta}}\right]^{\frac{t}{\alpha-1}} \qquad (4.18)$$

On differentiating (4.18) w.r.t. t at $t = 0$, we get,

$$\frac{\partial}{\partial t} I\alpha(P/Q\,;T) = (\alpha-1)^{-1} \left[\frac{\sum_{i=1}^{n} (p_i)^{\alpha\beta} (q_i)^{\alpha-1}}{\sum_{i=1}^{n} (p_i)^{\beta}}\right]^{\frac{t}{\alpha-1}} = H_{\alpha}^{\beta}(P/Q) \qquad (4.19)$$

ii) If we set $\beta = 1$ in (2.8) ,we get,

$$I\alpha(P/Q\,;T) = \left[\sum_{i=1}^{n} (p_i)^{\alpha} (q_i)^{\alpha-1}\right]^{\frac{t}{\alpha-1}} \qquad (4.20)$$

which is generalized relative information generating function of order α.

iii) In case $\beta = 1$, equation (4.15) reduces to

$$\frac{\partial}{\partial t} I\alpha(P/Q\,;U,T) = \left[\frac{\sum_{i=1}^{n} U_i p_i^{\alpha} q_i^{\alpha-1}}{\sum_{i=1}^{n} U_i p_i}\right]^{\frac{t}{\alpha-1}} \qquad (4.21)$$

which is the useful relative information generating function of order α.

On differentiating (4.21) gives

$\frac{\partial}{\partial t}I\alpha$(P/Q;U,T)=$(\alpha-1)^{-1}log\left[\frac{\sum_{i=1}^{n} U_i p_i{}^{\alpha} q_i{}^{\alpha-1}}{\sum_{i=1}^{n} U_i p_i}\right]$=H$\alpha$(P/Q;U) (4.22)
which is the new generalized relative information.
Further if $\alpha \to 1$, equation (4.22) becomes
H(P/Q; U) = $\left[\frac{\sum_{i=1}^{n} U_i p_i{}^{\alpha} q_i{}^{\alpha-1}}{\sum_{i=1}^{n} U_i p_i}\right]$ (4.23)
When U_i=1, then (4.23) reduces to Kullback-Lieblers measure of relative information which is given as:
H(P/Q ; U) =$\frac{\sum_{i=1}^{n} p_i log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} p_i}$
Since $\sum_{i=1}^{n} p_i$=1, then
H(P/Q ; U) = $\sum_{i=1}^{n} p_i log\left(\frac{p_i}{q_i}\right)$ which is similar to equation (4.1).

5. Two generalized 'useful' relative information function for somep probability distributions

Example 5.1: For the Uniform probability distribution $\left(\frac{1}{N},\frac{1}{N},\dots,\frac{1}{N}\right)$ and Uniform Utility distribution $(u,u,\dots,u)$ and predicted probability distribution$\left(\frac{1}{M},\frac{1}{M},\dots,\frac{1}{M}\right)$
Also put β = 1 in equation (4.6), we get:

$$I_{\alpha,\beta}(P/Q,t) = \left[\frac{\sum_{i=1}^{n} u\left(\frac{1}{N}\right)^{\alpha}\left(\frac{1}{M}\right)^{\alpha-1}}{\sum_{i=1}^{n} u\left(\frac{1}{N}\right)}\right]^{\frac{t}{\alpha-1}}$$

$$= \left[\frac{\left(\frac{1}{N}\right)^{\alpha}\left(\frac{1}{M}\right)^{\alpha-1}}{\left(\frac{1}{N}\right)}\right]^{\frac{t}{\alpha-1}}$$

$$I_{\alpha,\beta}(P/Q,t)= \left(\frac{1}{NM}\right)^{t}$$

From equation (4.7) we get,

$$H_{\alpha}{}^{\beta}(\text{P/Q ; t}) = \frac{1}{\alpha-1} log\left[\frac{\sum_{i=1}^{n} u\left(\frac{1}{N}\right)^{\alpha}\left(\frac{1}{M}\right)^{\alpha-1}}{\sum_{i=1}^{n} u\left(\frac{1}{N}\right)}\right]$$

$$H_{\alpha}{}^{\beta}(\text{P/Q ; t}) = -[log\, N + log M]$$

Example 5.2: Geometric distribution: Consider the Geometric distribution (q, qp, qp^2,....), (z, zw, zw^2,....); z+w=1, p+q=1 and Geometric utility distribution (v, vu, vu^2,....). It is the most general case when utility also follows geometric distribution. Then from equation (1.6) after substituting β=1,
Iα,β(P;U;t)=

$$\left[\sum_{i=1}^{n}(v, v\text{u}, v\text{u}^2, \dots)\,(q, q\text{p}, q\text{p}^2, \dots)^{\alpha}(\text{z}, \text{zw}, \text{zw}^2)^{\alpha-1} \Big/ \sum_{i=1}^{n}(v, v\text{u}, v\text{u}^2, \dots)\,(q, q\text{p}, q\text{p}^2, \dots)\right]^{t/\alpha-1}$$

$$= zqv\left[\sum_{i=1}^{n}(1, \text{u}, \text{u}^2, \dots)\,(1, \text{p}, \text{p}^2, \dots)^{\alpha}(1, \text{w}, \text{w}^2)^{\alpha-1} \Big/ qv\sum_{i=1}^{n}(1, \text{u}, \text{u}^2, \dots)\,(1, \text{p}, \text{p}^2, \dots)\right]^{t/\alpha-1}$$

$$= z\left[\left(\frac{1}{1-u}\right)\left(\frac{1}{1-p}\right)^{\alpha}\left(\frac{1}{1-w}\right)^{\alpha-1} \Big/ \left(\frac{1}{1-u}\right)\left(\frac{1}{1-p}\right)\right]^{t/\alpha-1}$$

$\Rightarrow I_{\alpha,\beta}(P/Q,t)= Z\left(\frac{1}{Zq}\right)^{t}$

Also from equation (4.7) and put $\beta = 1$, we get,

$$\frac{\partial}{\partial t} I_{\alpha,\beta}(P/Q,t)= (\alpha-1)^{-1} log\left[\frac{\sum_{i=1}^{n}(v,vu,vu^2,\ldots)(q,qp,qp^2,\ldots)^{\alpha}(Z,ZW,\ldots)^{\alpha-1}}{\sum_{i=1}^{n}(v,vu,vu^2,\ldots)(q,qp,qp^2,\ldots)}\right]$$

$$= \frac{1}{\alpha-1} log\left[\frac{vqz\left(\frac{1}{1-U}\right)\left(\frac{1}{1-p}\right)^{\alpha}\left(\frac{1}{1-W}\right)^{\alpha-1}}{vq\left(\frac{1}{1-U}\right)\left(\frac{1}{1-p}\right)}\right]$$

$$= z\frac{1}{\alpha-1} log\left[\left(\frac{1}{Zq}\right)^{\alpha-1}\right]$$

$$\Rightarrow \frac{\partial}{\partial t} I_{\alpha,\beta}(P/Q,t) = -zlogz - zlogq .$$

Example 5.3 : For exponential distribution we consider

$p(x)=\lambda e^{-\lambda x}$, $\lambda> 0$, $0\leq x\leq \infty$

$u(y)=ue^{-uy}$, $u>0$, $0\leq y \leq \infty$

$q(z)=\gamma e^{-\gamma z}$, $\gamma>0$, $0\leq \gamma\leq \infty$

From equation (4.6) after putting β=1 we get

$$I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \left[\frac{\sum_{i=1}^{n}(ue^{-uy})\left(\lambda e^{-\lambda x}\right)^{\alpha}(\gamma e^{-\gamma z})^{\alpha-1}}{\sum_{i=1}^{n}\lambda e^{-\lambda x}\, ue^{-uy}}\right]^{\frac{t}{\alpha-1}}$$

$$= \left[\frac{\mu\sum_{i=1}^{n}(ue^{-uy})\left(\lambda e^{-\lambda x}\right)^{\alpha}(\gamma e^{-\gamma z})^{\alpha-1}}{\lambda\mu\sum_{i=1}^{n} e^{-\lambda x}\, e^{-uy}}\right]^{\frac{t}{\alpha-1}}$$

$$= \left[\left(\lambda e^{-\lambda x}\right)^{\alpha-1}(\gamma e^{-\gamma z})^{\alpha-1}\right]^{\frac{t}{\alpha-1}}$$

$$I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \left[\lambda\gamma e^{-\lambda x-\gamma z}\right]^{t}$$

Also from equation (4.7), we have,

$$\frac{\delta}{\delta t} I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n}(ue^{-uy})(\lambda e^{-\lambda x})^{\alpha}(\gamma e^{-\gamma z})^{\alpha-1}}{\sum_{i=1}^{n}\lambda e^{-\lambda x}\, ue^{-uy}}\right]^{\frac{t}{\alpha-1}}$$

$$= \frac{1}{\alpha-1}\log\left[\left(\lambda e^{-\lambda x}\right)(\gamma e^{-\gamma z})^{\alpha-1}\right]$$

$$\frac{\delta}{\delta t} I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \log\lambda - \lambda x + \log\gamma - \gamma z$$

Example 5.4: For poisson distribution, we consider :

$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$; x=0,1,2,. . . ;$\lambda> 0$

$U(y)=\frac{e^{-\mu}\mu^{y}}{y!}$; y=0,1,2,. . . ;$\mu> 0$

$Q(z)=\frac{e^{-\gamma}\gamma^{z}}{z!}$; z =0,1,2,. . . ;$\gamma> 0$

After substituting these values and β=1 in (4.6) we get

$$I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \left[\frac{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)^{\alpha}\left(\frac{e^{-\gamma}\gamma^{z}}{z!}\right)^{\alpha-1}}{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)}\right]^{t/\alpha-1}$$

$$=\left[\sum_{i=1}^{n}\left(\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)\right)^{\alpha-1}\left(\frac{e^{-\gamma}\gamma^{z}}{z!}\right)^{\alpha-1}\right]^{t/\alpha-1}$$

$$I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \left[\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\left(\frac{e^{-\gamma}\gamma^{z}}{z!}\right)\right)\right]^{t}$$

Also from equation (4.7) we get

$$H_{\alpha}^{\beta}\left(\frac{P}{Q};U\right)=\frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)^{\alpha}\left(\frac{e^{-\gamma}\gamma^{z}}{z!}\right)^{\alpha-1}}{\sum_{i=1}^{n}\left(\frac{e^{-\mu}\mu^{y}}{y!}\right)\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)}\right]$$

$$=\frac{1}{\alpha-1}\log\left[\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)\left(\frac{e^{-\gamma}\gamma^{z}}{z!}\right)^{\alpha-1}\right]$$

$$H_{\alpha}^{\beta}\left(\frac{P}{Q};U\right) =\log\frac{e^{-\lambda}\lambda^{x}}{x!} + \log\frac{e^{-\gamma}\gamma^{z}}{z!}$$

Example 5.5: Gamma distribution with one parameter λ, μ & γ

Consider $p(x) =\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}, \lambda> 0,\ 0 < x < \infty$

$U(y)=\frac{e^{-y}y^{\mu-1}}{\Gamma\mu}, \mu> 0,\ 0 < y < \infty$

$Q(z)=\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}, \gamma> 0,\ 0 < z < \infty$

Put these values in equation (4.6) and β_i=β=1 we get

$$I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \left[\frac{\sum_{i=1}^{n}\left(\frac{e^{-y}y^{\mu-1}}{\Gamma\mu}\right)\left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)^{\alpha}\left(\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}\right)^{\alpha-1}}{\sum_{i=1}^{n}\frac{e^{-y}y^{\mu-1}}{\Gamma\mu}\left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)}\right]^{t/\alpha-1}$$

$$=\left[\sum_{i=1}^{n}\left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)^{\alpha}\left(\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}\right)^{\alpha-1}\right]^{t/\alpha-1}$$

$$I_{\alpha,\beta}\left(\frac{P}{Q};t\right) = \left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}\right)^{t}$$

Also from equation (4.7), we have,

$$= \frac{1}{\alpha-1}\log\left[\frac{\sum_{i=1}^{n}e^{-y}y^{\mu-1}\left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)^{\alpha}\left(\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}\right)^{\alpha-1}}{\sum_{i=1}^{n}\left(\frac{e^{-y}y^{\mu-1}}{\Gamma\mu}\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)}\right]$$

$$=\frac{1}{\alpha-1}\log\left[\left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)\left(\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}\right)^{\alpha-1}\right]^{\alpha-1}$$

$$H_{\alpha}^{\beta}\left(\frac{P}{Q};U\right)=\log\left[\left(\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}\right)\left(\frac{e^{-z}z^{\gamma-1}}{\Gamma\gamma}\right)\right]$$

6. Conclusion

We have shown that parametric information generation functions defined in this paper are generalization of the information generating function based on Shannon's measure of Information.

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