SOME IMPROVED SUGGESTIONS IN CIRCULAR SYSTEMATIC SAMPLING UNDER FULLY AND PARTIALLY RESPONDENT CASE

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Abstract

This paper considers the problem of estimating population mean $\overline{Y}$ of survey variable Y using circular systematic sampling design along with non response problem. Motivated by Sahai and Ray (1980), Koyunsu and Kadilar (2010), Singh and Solanki (2013) and Singh and Malik(2014), we have suggested three modified class of estimators for $\overline{Y}$. The properties of the suggested estimators are discussed and are compared with sample mean and linear regression estimator. Also, an empirical study is carried out to judge the merits of suggested estimators over other competitors based on circular systematic sampling over simple random sampling.

Key Words: Regression Estimator, Circular Systematic Sampling, Two-Phase Sampling, Non- Response, Asymptotic Variance, Efficiency.

1. Introduction

In sample surveys, auxiliary information on the finite population under study is quite often available from previous experience, census or administrative databases. The sampling literature describes a wide variety of techniques for using auxiliary information to improve the sampling design and/or obtain more efficient estimators. It is well known that when the auxiliary information is to be used at the estimation stage, the ratio, product and regression methods are widely employed.

Systematic sampling technique is operationally more convenient than simple random sampling. It also ensure at the same time each unit has equal probability of inclusion in the sample. In this method of sampling, the first unit is selected with the help of random number and remaining units are selected automatically according to predetermined pattern. Hasel (1942) and Griffth(1945-1946) found systematic sampling to be efficient and convenient in sampling certain natural populations like forest areas for estimating the volume of the timber and area under different types of cover. Cochran (1946) and Hajeck (1959) had stated that in large-scale sampling work, this procedure provides more efficient estimators than those provided by simple random sampling and/or stratified random sampling for certain populations.

When the population mean $\overline{X}$ of the auxiliary variable x is known, Swain(1964) and Shukla (1971) have suggested the ratio and product estimators for the population mean $\overline{Y}$ of the survey variable y, respectively, along with their properties in systematic sampling. Some other notable work in this area are Singh and Singh (1998), Singh R. et al.(2012), Singh and Jatwa (2012), Singh and Solanki(2012) and Verma et al. (2014).

In linear systematic sampling, given a sample size n, sampling is possible only if population size N is divisible by n. Even when this condition is satisfied, the scheme cannot provide estimate of variance of the sample mean. This scheme has two drawbacks namely, given N, n has limited choice and variance of the sample mean is not estimable. The first limitation could be removed through circular systematic sampling as suggested by Lahiri(1952). The procedure consists in selecting a unit, by a random start, from 1 to N and then thereafter selecting every k^{th} unit, k being an integer nearest to N/n, in a circular manner, until a sample of n units is obtained. Suppose that a unit with random number i is selected. The sample will then consists of the units corresponding to the serial numbers

$$
Label = \begin{cases} i + jk, & 1 \le i + jk \le N, \\ & \qquad \qquad for\ i = 0,1,.......,(n-1) \\ i + jk - N, & N < i + jk. \end{cases}
$$

[for details see Singh and Chaudhary (1986),pp83].

In the following manner, we may draw N circular systematic samples, each of size n as displayed in Table 1.

Sample number					
1	2	......	i	....	N
u_1	u_2		u_i		u_N
u_{k+1}	u_{k+2}		u_{k+i}		u_{2N}
....	....		....		....
....	....		....		....
$u_{(n-1)k+1}$	$u_{(n-1)k+2}$		$u_{(n-1)k+i}$		u_{nN}

Table 1: Possible Samples using Circular Systematic Sampling

From these N possible samples, a sample of size n is selected randomly to observe Y and X.

The present problem aims to give some contribution on this subject. For this purpose, taking motivation from Sahai and Ray (1980), Koyunsu and Kadilar (2010), Singh and Solanki (2013) and Singh and Malik (2014), three modified classes of estimators are suggested for the population mean using auxiliary information in circular systematic sampling design, following this we have also studied the effect of non-response (present in study variable y) on suggested estimators.

2. Processing and notations used in circular systematic sampling

Let us suppose that U^* be a finite population consists of N distinct labelled units i.e. $U^* = (U_1, U_2 \ldots\ldots\ldots .U_N)$ and n be a fixed sample size.

Also, let Y and X be study and auxiliary variables taking values y_{ij} and x_{ij} $i = (1,2,\ldots, N), j = (1,2,\ldots, n)$.

The CSS sample means $\overline{y}_{CSS} = \sum_{j-1}^{n} y_{ij} / n$ and $\overline{x}_{CSS} = \sum_{j-1}^{n} x_{ij} / n$ are unbiased estimates of population means $\overline{Y}_{CSS} = \sum_{j-1}^{N} y_{ij} / N$ and $\overline{X}_{CSS} = \sum_{j-1}^{N} X_{ij} / N$ respectively.

The variance of $\overline{y}_{CSS}$ and $\overline{x}_{CSS}$ under CSS design is written as-

$$V(\overline{y}_{CSS}) = \theta_1 \left[1 + (n-1)\rho_y \right] \frac{S_y^2}{n} = \widetilde{S}_y^2 = \overline{Y}^2 \widetilde{C}_y^2$$

and

$$V(\overline{x}_{CSS}) = \theta_1 \left[1 + (n-1)\rho_x \right] \frac{S_x^2}{n} = \widetilde{S}_x^2 = \overline{X}^2 \widetilde{C}_x^2$$

where

$$\theta_1 = \left(\frac{N-1}{N} \right), \quad S_y^2 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \overline{Y})^2}{n(N-1)}, \quad S_x^2 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (x_{ij} - \overline{X})^2}{n(N-1)}$$

With

$$\rho_y = \frac{2}{n(n-1)(N-1)S_y^2} \sum_{i=1}^{N} (y_{ij} - \overline{Y})(y_{iu} - \overline{Y})$$

and

$$\rho_x = \frac{2}{n(n-1)(N-1)S_x^2} \sum_{i=1}^{N} (x_{ij} - \overline{X})(x_{iu} - \overline{X})$$

where (ρ_y, ρ_x) represents intraclass correlation coefficients between pairs of units within the CSS for variable Y and X, respectively.
Also,

$$\text{Cov}(\overline{y}_{CSS}, \overline{x}_{CSS}) = \theta_1 \left[1 + (n-1)\rho_y\right]^{1/2} \left[1 + (n-1)\rho_x\right]^{1/2} \frac{S_{yx}}{n} = \widetilde{S}_{yx} = \overline{Y}\,\overline{X}\widetilde{C}_{yx}$$

where

$$S_{yx} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{n}(y_{ij} - \overline{Y})(x_{ij} - \overline{X})}{n(N-1)}.$$

3. Estimators in literature

It is well known that regression estimators are always more efficient then usual mean and ratio estimators, at least asymptotically, thus we consider linear regression estimators based on CSS as standard result for making comparison with our suggested class of estimators.

The linear regression estimator of the population mean $\overline{Y}$ based on CSS with known $\overline{X}$ is defined as

$$\overline{y}_{1r}(c) = \overline{y}_{CSS} + \hat{\beta}_{yx}(\overline{X} - \overline{x}_{CSS}), \qquad (1)$$

where $\hat{\beta}_{yx} = \dfrac{s_{yx}}{s_x^2}$ is an estimator for population regression coefficient β_{yx} with

$$s_x^2 = \frac{\sum_{j=1}^{n}(x_{ij} - \overline{x}_{CSS})^2}{(n-1)} \text{ and } s_{yx} = \frac{\sum_{j=1}^{n}(y_{ij} - \overline{x}_{CSS})(x_{ij} - \overline{y}_{CSS})}{(n-1)}$$

The asymptotic variance of $\overline{y}_{1r}(c)$ is respectively given by

$$AV\left[\overline{y}_{1r}(c)\right] = \widetilde{S}_y^2 \left(1 - \widetilde{\rho}_{yx}^2\right), \qquad (2)$$

where, $\widetilde{\rho}_{yx}^2 = \dfrac{\widetilde{S}_{yx}^2}{\widetilde{S}_y^2 \widetilde{S}_x^2}$

When population mean $\overline{X}$ is unknown, double sampling scheme is used. Under double sampling scheme, first we divide the population into N clusters of size n, each according to CSS, and select randomly m distinct clusters (1< m <k) to estimate $\overline{X}$ only. In second phase, a cluster is selected randomly from m CSSs to estimate $\overline{Y}$. Hence, the expression for $\overline{y}_{1r}(c)$, with unknown $\overline{X}$, is given as

$$\overline{y}'_{1r}(c) = \overline{y}_{CSS} + \hat{\beta}_{yx}(\overline{x}'_{CSS} - \overline{x}_{CSS}), \qquad (3)$$

where $\overline{x}'_{CSS} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}$

Define,

$$V(\overline{x}') = \text{Cov}(\overline{x}'_{CSS}, \overline{x}_{CSS}) = \theta_1 \left[1 + (n-1)\rho_x\right] \frac{S_x^2}{nm} = \frac{\widetilde{S}_x^2}{m}$$

and

$$\text{Cov}[\bar{y}_{CSS}] = \theta_1 [1 + (n-1)\rho_y]^{1/2} [1 + (n-1)\rho_x]^{1/2} \frac{S_{yx}}{nm} = \frac{\tilde{S}_{yx}}{m}$$

The asymptotic variance of $\bar{y}'_{lr}\,(c)$ is given by

$$AV(\bar{y}'_{lr}\,(c)) = \tilde{S}_y^2 \left(1 - \tilde{\tilde{\rho}}_{yx}^2\right) \tag{4}$$

where, $\tilde{\tilde{\rho}}_{yx}^2 = \dfrac{\tilde{\tilde{S}}_{yx}^2}{\tilde{S}_y^2 \tilde{\tilde{S}}_x^2}$, such that

$$\left.\begin{array}{l} \tilde{\tilde{S}}_{yx} = \left(\dfrac{m-1}{m}\right)\tilde{S}_{yx} \text{ and } \tilde{\tilde{S}}_x^2 = \left(\dfrac{m-1}{m}\right)\tilde{S}_x^2 \\[2ex] \text{with} \\[2ex] \tilde{\tilde{C}}_x^2 = \dfrac{\tilde{\tilde{S}}_x^2}{\overline{Y}^2} \text{ and } \tilde{\tilde{C}}_{yx} = \dfrac{\tilde{\tilde{S}}_{yx}}{\overline{Y}\,\overline{X}} \end{array}\right\} \tag{5}$$

4. Suggested family of estimators

Much literature has been written on sampling from finite populations to address the issue of the efficient estimation of the mean (or total) of a survey variable when auxiliary variables are available. Our analysis refers to simple random sampling without replacement (SRSWOR) and circular systematic sampling (CSS) considers, for brevity, the case when only a single auxiliary variable is used.

In this section, we have suggested three modified class of estimators for estimating $\overline{Y}$ using prior knowledge on x, based on SRSWOR and circular systematic sampling. The suggested modified classes are motivated from Singh and Solanki (2013), Singh and Malik (2014), Sahai and Ray (1980) and Koyuncu and Kadilar (2010]). After observing theoretical and empirical results, we infer that the suggested estimators performs better than usual linear regression estimators in both SRSWOR and CSS design, which demonstrate the superiority of suggested estimators. Also, we observed that CSS design is more convenient and efficient then SROWOR design.

4.1 Difference-type class of estimator

Motivated from Singh and Solanki (2013) and Koyuncu and Kadilar (2010), we propose the following difference-type class of estimators for estimating population mean $\overline{Y}$ of a survey variable under CSS assuming $\overline{X}$ is known

$$T_1 = \left[\kappa_1 \bar{y}_{CSS} + \kappa_2 \bar{x}^*_{CSS} + (1 - \kappa_1 - \kappa_2)\overline{X}^*\right]\exp\left[\alpha\,\frac{\overline{X}^* - \bar{x}^*_{CSS}}{\overline{X}^* + \bar{x}^*_{CSS}}\right] \tag{6}$$

where (κ_1, κ_2) are suitably choosen scalars to be properly determined for minimum mean square error (MSE) of suggested estimators, $\overline{x}^*_{CSS} = \eta \overline{x}_{CSS} + \lambda$, $\overline{X}^* = \eta \overline{X} + \lambda$ with (η, λ) are either constants or function of some known population parameters and α being constants which take finite values for designing the different estimators (see Singh and Kumar(2011)).

To obtain the bias and *MSE* expressions of the proposed class of estimators T_1, we define

$$\delta_y = \frac{\left(\overline{y}_{css} - \overline{Y}\right)}{\overline{Y}} \text{ and } \delta_x = \frac{\left(\overline{x}_{css} - \overline{X}\right)}{\overline{X}}$$

Such that

$$\left.\begin{array}{l} E(\delta_y) = E(\delta_x) = 0 \\[2mm] E(\delta_y^2) = \tilde{C}_y^2, E(\delta_x^2) = \tilde{C}_x^2 \text{ and } E(\delta_y \delta_x) = \tilde{C}_{yx} \end{array}\right\} \tag{7}$$

The expressions for asymptotic bias and asymptotic variance of the suggested estimator T_1 using CSS are given respectively as

$$AB(T_1) = \left[\phi + \phi_1 \tilde{C}_x^2 + \kappa_1 \left\{ A_2 \phi_2 - A - \frac{\overline{Y} \alpha \tau_1 \tilde{C}_{yx}}{2} \right\} - \kappa_2 \frac{\eta \overline{X} \alpha \tau_1 \tilde{C}_x^2}{2} \right] \tag{8}$$

and

$$AV(T_1) = \left[N + \kappa_1^2 N_{11} + \kappa_2^2 N_{12} - 2\kappa_1 N_{13} - 2\kappa_2 N_{14} + 2\kappa_1 \kappa_2 N_{15} \right] \tag{9}$$

Minimisation of $AV(T_1)$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_1^* = \left(\frac{N_{12} N_{13} - N_{14} N_{15}}{N_{11} N_{12} - N_{15}^2} \right) \text{ and } \kappa_2^* = \left(\frac{N_{11} N_{14} - N_{13} N_{15}}{N_{11} N_{12} - N_{15}^2} \right)$$

where

$$N = \phi^2 + \frac{\alpha^2 \tau_1^2 \overline{X}^{*2} \tilde{C}_x^2}{2} + 2\phi \phi_1 \tilde{C}_x^2,$$

$$N_{11} = \overline{Y}^2 \tilde{C}_y^2 + \phi^2 + \frac{\alpha^2 \tau_1^2 \phi^2}{4} \tilde{C}_x^2 - 2\phi \phi_2 \tilde{C}_x^2 + 2\alpha \tau_1 \phi \overline{Y} \tilde{C}_{yx}$$

$$N_{12} = \eta^2 \overline{X}^2 \tilde{C}_x^2,$$

$$N_{13} = \phi^2 + \frac{\alpha \tau_1 \overline{Y}(\phi + \overline{X}^*)}{2} \tilde{C}_{yx} + \frac{\alpha^2 \tau_1^2 \overline{X}^* \phi}{2} \tilde{C}_x^2 + \phi \tilde{C}_x^2 (\phi_1 - \phi_2)$$

$$N_{14} = \eta \alpha \tau_1 \overline{X}(\phi + \overline{X}^*) \tilde{C}_x^2 / 2, N_{15} = \eta \alpha \tau_1 \phi \overline{X} \tilde{C}_x^2 + \eta \overline{X} \overline{Y} \tilde{C}_{yx}$$

Such that, $\tau_1 = \dfrac{\eta \overline{X}}{\eta \overline{X} + \lambda}$, $\phi = \left(\overline{X}^* - \overline{Y}\right)$, $\phi_1 = \left(\dfrac{\alpha \tau_1^2 \overline{X}^*}{4} + \dfrac{\alpha^2 \tau_1^2 \overline{X}^*}{8} \right)$

and $\phi_2 = \left(\dfrac{\alpha\tau_1^2\overline{Y}}{4} + \dfrac{\alpha^2\tau_1^2\overline{Y}}{8} - \dfrac{\alpha\tau_1^2\overline{X}^*}{4} - \dfrac{\alpha^2\tau_1^2\overline{X}^*}{8} \right)$.

Now suppose $\overline{X}$ is unknown, the analogue of T_1 becomes

$$T_1' = \left[\kappa_{d1}\overline{y}_{CSS} + \kappa_{d2}\overline{x}_{CSS}^* + (1 - \kappa_{d1} - \kappa_{d2})\overline{x}_{CSS}^{'*} \right] \exp\left[\alpha\,\frac{\overline{x}_{CSS}^{'*} - \overline{x}_{CSS}^*}{\overline{x}_{CSS}^{'*} + \overline{x}_{CSS}^*} \right] \qquad (10)$$

where, the notations used here are already defined earlier.

To obtain the bias and *MSE* of the proposed class of estimators T_1, we define

$$\delta_x' = \frac{\left(\overline{x}_{CSS}' - \overline{X}\right)}{\overline{X}}$$

Such that

$$\left. E(\delta_x') = 0, \quad E(\delta_x'^2) = E(\delta_x', \delta_x) = \frac{\widetilde{C}_x^2}{m} \quad \text{and} \quad E(\delta_y\delta_x') = \frac{\widetilde{C}_{yx}}{m} \right\} \qquad (11)$$

The expressions for asymptotic bias and asymptotic variance of the suggested estimator T_1' using CSS are given respectively as

$$AB(T_1') = \left[\phi + \phi_1\widetilde{C}_x^2 + \kappa_1\left\{ A_2\phi_2 - A - \frac{\overline{Y}\alpha\tau_1\widetilde{C}_{yx}}{2} \right\} - \kappa_2\,\frac{\eta\overline{X}\alpha\tau_1\widetilde{C}_x^2}{2} \right] \qquad (12)$$

and

$$AV(T_1') = \left[N + \kappa_{d1}^2 N_{11} + \kappa_{d2}^2 N_{12} - 2\kappa_{d1}N_{13} - 2\kappa_{d2}N_{14} + 2\kappa_{d1}\kappa_{d2}N_{15} \right] \qquad (13)$$

Minimisation of $AV(T_1')$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_{d1}^* = \left(\frac{N_{12}N_{13} - N_{14}N_{15}}{N_{11}N_{12} - N_{15}^2} \right) \quad \text{and} \quad \kappa_{d2}^* = \left(\frac{N_{11}N_{14} - N_{13}N_{15}}{N_{11}N_{12} - N_{15}^2} \right)$$

where,

$$N = \phi^2 + \frac{\alpha^2\tau_1^2\overline{X}^{*2}\widetilde{C}_x^2}{2} + 2\phi\phi_1\widetilde{C}_x^2,$$

$$N_{11} = \overline{Y}^2\widetilde{C}_y^2 + \phi^2 + \frac{\alpha^2\tau_1^2\phi^2}{4}\widetilde{C}_x^2 - 2\phi\phi_2\widetilde{C}_x^2 + 2\alpha\tau_1\phi\overline{Y}\widetilde{C}_{yx}$$

$$N_{12} = \eta^2\overline{X}^2\widetilde{C}_x^2,$$

$$N_{13} = \phi^2 + \frac{\alpha\tau_1\overline{Y}(\phi + \overline{X}^*)}{2}\widetilde{\widetilde{C}}_{yx} + \frac{\alpha^2\tau_1^2\overline{X}^*\phi}{2}\widetilde{\widetilde{C}}_x^2 + \phi\widetilde{C}_x^2(\phi_1 - \phi_2)$$

$$N_{14} = \eta\alpha\tau_1\overline{X}\left(\phi + \overline{X}^*\right)\widetilde{\widetilde{C}}_x^2 \Big/ 2, \ N_{15} = \eta\alpha\tau_1\phi\overline{X}\widetilde{\widetilde{C}}_x^2 + \eta\overline{XY}\widetilde{\widetilde{C}}_{yx}$$

with, $\widetilde{\widetilde{C}}_x^2$ and $\widetilde{\widetilde{C}}_{yx}$ which is already defined in equation (4).

Note 1: It can be observed from equation (8), (9) and (12), (13) that the asymptotic bias and minimum asymptotic variance of T_1 and $T_1^{'}$ look similar. However, due to single and double sampling design the dissimilarity exists only in terms $\left(\widetilde{C}_x^2, \widetilde{C}_{yx}\right)$ and $\left(\widetilde{\widetilde{C}}_x^2, \widetilde{\widetilde{C}}_{yx}\right)$.

4.2 Exponential- ratio type class of estimators

Motivated by Sahai and Ray(1980)and Singh and Malik (2014), the following modified class of estimators has been suggested for the population mean $\overline{Y}$ assuming that $\overline{X}$ is known

$$T_2 = \overline{y}_{CSS}\left[\kappa_1 + \kappa_2\left\{2 - \frac{\overline{x}_{CSS}}{\overline{X}}\right\}^\beta\right]\exp\left[\gamma\frac{(\mu\overline{X} + v) - (\mu\overline{x}_{CSS} + v)}{(\mu\overline{X} + v) + (\mu\overline{x}_{CSS} + v)}\right] \qquad (14)$$

where β, μ and v are either real numbers or function of known parameters of auxiliary variable x such as S_x, C_x, ρ_x etc. The scalars γ takes values -1,0 and -1 for ratio and product type estimators, respectively.

The expressions for asymptotic bias and asymptotic variance of the suggested estimator T_2 using CSS are given as

$$AB(T_2) = \overline{Y}\left[\kappa_1\left\{1 + \phi_3\widetilde{C}_x^2 - \frac{\gamma\tau_2\widetilde{C}_{yx}}{2}\right\} + \kappa_2\left\{1 + \phi_4\widetilde{C}_x^2 - \phi_5\widetilde{C}_{yx}\right\} - 1\right] \qquad (15)$$

and

$$AV(T_2) = \overline{Y}^2\left[\kappa_1^2 N_{21} + \kappa_2^2 N_{22} - 2\kappa_1 N_{23} - 2\kappa_2 N_{24} + 2\kappa_1\kappa_2 N_{25}\right] \qquad (16)$$

Minimisation of $AV(T_2)$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_1^* = \left(\frac{N_{22}N_{23} - N_{24}N_{25}}{N_{21}N_{22} - N_{25}^2}\right) \text{ and } \kappa_2^* = \left(\frac{N_{21}N_{24} - N_{23}N_{25}}{N_{21}N_{22} - N_{25}^2}\right)$$

where

$$N_{21} = \left[1 + \widetilde{C}_y^2 + \left\{\frac{\gamma^2\tau_2^2}{4} + 2\phi_3\right\}\widetilde{C}_x^2 - 2\gamma\tau_2\widetilde{C}_{yx}\right],$$

$$N_{22} = \left[1 + \widetilde{C}_y^2 + \left\{\phi_5 + 2\phi_4\right\}\widetilde{C}_x^2 - 4\phi_5\widetilde{C}_{yx}\right],$$

$$N_{23} = \left[1 + \phi_3\widetilde{C}_x^2 - \frac{\gamma\tau_2\widetilde{C}_{yx}}{2}\right], \ N_{24} = \left[1 + \phi_4\widetilde{C}_x^2 - \phi_5\widetilde{C}_{yx}\right]$$

$$N_{25} = \left[1 + \widetilde{C}_y^2 + \phi_4 \widetilde{C}_x^2 - 2\phi_5 \widetilde{C}_{yx} - \beta\tau_2 \widetilde{C}_{yx} + \frac{\gamma\tau_2 \phi_5 \widetilde{C}_x^2}{2} + \phi_3 \widetilde{C}_x^2 \right]$$

Such that,

$$\tau_2 = \frac{\mu\overline{X}}{\mu\overline{X} + \nu}, \quad \phi_3 = \left(\frac{\gamma\tau_2^2}{4} + \frac{\gamma^2\tau_2^2}{8} \right), \quad \phi_4 = \left(\frac{\gamma\tau_2^2}{4} + \frac{\gamma^2\tau_2^2}{8} + \frac{\beta\tau_2\gamma}{2} - \frac{\beta(\beta-1)}{2} \right)$$

and $\phi_5 = \frac{\gamma\tau_2}{2} + \beta$.

Now assuming that $\overline{X}$ is unknown, then the analogue of T_2 becomes

$$T_2' = \overline{y}_{CSS} \left[\kappa_1 + \kappa_2 \left\{ 2 - \frac{\overline{x}_{CSS}}{\overline{x}'_{CSS}} \right\}^\beta \right] \exp \left[\gamma \frac{(\mu\overline{x}'_{CSS} + \nu) - (\mu\overline{x}_{CSS} + \nu)}{(\mu\overline{x}'_{CSS} + \nu) + (\mu\overline{x}_{CSS} + \nu)} \right] \qquad (17)$$

The asymptotic bias and variance of T_2' will be almost similar to T_2. The difference between the AB and AV of T_2' and T_2 will be same like the difference between T_1' and T_1 explained in section **4.1**. Therefore, replacing the terms $\left(\widetilde{C}_x^2, \widetilde{C}_{yx} \right)$ in (15) and (16) by and $\left(\widetilde{\widetilde{C}}_x^2, \widetilde{\widetilde{C}}_{yx} \right)$, we get the asymptotic bias and minimum asymptotic variance of T_2'.

4.3 Generalised exponential-ratio type estimators

Motivated by Sahai and Ray (1980)and Koyuncu and Kadilar (2010), the following modified class of estimators has been defined for the population mean $\overline{Y}$ assuming that $\overline{X}$ is known

$$T_3 = \overline{y}_{CSS} \left[\kappa_1 \left\{ 2 - \frac{\overline{x}_{CSS}}{\overline{X}} \right\}^{\beta_1} + \kappa_2 \exp \left\{ \frac{\eta_1 \left(\overline{X} - \overline{x}_{CSS} \right)}{\left(\overline{X} + \overline{x}_{CSS} \right) + 2\lambda_1} \right\} \right] \qquad (18)$$

where ($\kappa_1, \kappa_2, \beta_1$) are defined earlier and (η_1, λ_1) being constants, which take finite real values or faction of known parameters for designing the different family of proposed estimators.

The expressions for asymptotic bias and asymptotic variance of the suggested estimator T_3 using CSS are given as

$$AB(T_3) = \overline{Y} \left[\kappa_1 \left\{ 1 + \frac{\beta_1(\beta_1 - 1)}{2} \tilde{C}_x^2 - \beta_1 \tilde{C}_{yx} \right\} + \kappa_2 \left\{ 1 + \frac{3\tau_3^2}{8} \tilde{C}_x^2 - \frac{\tau_3}{2} \tilde{C}_{yx} \right\} - 1 \right] \qquad (19)$$

and

$$AV(T_3) = \overline{Y}^2 \left[\kappa_1^2 N_{31} + \kappa_2^2 N_{32} - 2\kappa_1 N_{33} - 2\kappa_2 N_{34} + 2\kappa_1\kappa_2 N_{35} \right] \qquad (20)$$

Minimisation of $AV(T_3)$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_1^* = \left(\frac{N_{32}N_{33} - N_{34}N_{35}}{N_{31}N_{32} - N_{35}^2} \right) \quad \text{and} \quad \kappa_2^* = \left(\frac{N_{31}N_{34} - N_{33}N_{35}}{N_{31}N_{32} - N_{35}^2} \right)$$

where

$$N_{31} = \left[1 + \widetilde{C}_y^2 + \beta_1 \widetilde{C}_x^2 - 4\beta_1 \widetilde{C}_{yx} \right],$$

$$N_{32} = \left[1 + \widetilde{C}_y^2 + \tau_3^2 \widetilde{C}_x^2 - 2\tau_3 \widetilde{C}_{yx} \right],$$

$$N_{33} = \left[1 - \frac{\beta_1(\beta_1 - 1)}{2} \widetilde{C}_x^2 - \beta_1 \widetilde{C}_{yx} \right], \quad N_{34} = \left[1 + \frac{3\tau_3^2}{8} \widetilde{C}_x^2 - \frac{\tau_3}{2} \widetilde{C}_{yx} \right]$$

$$N_{25} = \left[1 + \widetilde{C}_y^2 + \phi_6 \widetilde{C}_x^2 - 2\phi_7 \widetilde{C}_{yx} \right]$$

Such that, $\tau_3 = \dfrac{\eta_1 \overline{X}}{\eta_1 \overline{X} + \lambda_1}$, $\phi_6 = \dfrac{3\tau_3^2}{8} + \dfrac{\beta_1 \tau_3}{2} - \dfrac{\beta_1(\beta_1 - 1)}{2}$ and $\phi_7 = 2\beta_1 + \tau_3$

Now assuming that $\overline{X}$ is unknown, then the analogue of T_3 becomes

$$T_3^{'} = \overline{y}_{CSS} \left[\kappa_1 \left\{ 2 - \frac{\overline{x}_{CSS}}{\overline{x}'_{CSS}} \right\}^{\beta_1} + \kappa_2 \exp \left\{ \frac{\eta_1 (\overline{x}'_{CSS} - \overline{x}_{CSS})}{(\overline{x}'_{CSS} + \overline{x}_{CSS}) + 2\lambda_1} \right\} \right] \tag{21}$$

where $(\kappa_1, \kappa_2, \beta_1, \eta_1, \lambda_1)$ are already defined earlier.

Then the asymptotic bias and variance of $T_3^{'}$ can be the same like T_3. Because the difference between T_3 and $T_3^{'}$ will be the same like the difference which is in T_1 and $T_1^{'}$ i.e. the presence of $\left(\widetilde{\widetilde{C}}_x^2, \widetilde{\widetilde{C}}_{yx} \right)$ instead of $\left(\widetilde{C}_x^2, \widetilde{C}_{yx} \right)$.

There are many ways to construct the classes of proposed estimators T_1, T_2 and T_3. Many authors have discussed different family of estimators using known parameters and it is observed that, it increases the efficiency of the estimators. Thus we have given some members of the classes (T_1, T_2, T_3) using known population parameters.

Estimators	Constants		
T_1	η	λ	α
$T_{11} = \left[\kappa_1 \overline{y}_C + \kappa_2 (\overline{x}_C - N) + (1 - \kappa_1 - \kappa_2)(\overline{X} - N) \right] \exp\left[C_x \dfrac{\overline{x}_C - \overline{X}}{\overline{x}_C + \overline{X} - 2N} \right]$	1	-N	$-C_x$
$T_{12} = \left[\kappa_1 \overline{y}_C + \kappa_2 (\overline{x}_C - S_x) + (1 - \kappa_1 - \kappa_2)(\overline{X} - S_x) \right] \exp\left[C_x^2 \dfrac{\overline{x}_C - \overline{X}}{\overline{x}_C + \overline{X} - 2S_x} \right]$	1	$-S_x$	C_x^2
$T_{13} = \left[\kappa_1 \overline{y}_C - \kappa_2 (\overline{x}_C - S_x) - (1 - \kappa_1 - \kappa_2)(\overline{X} - S_x) \right] \exp\left[\dfrac{\overline{x}_C - \overline{X}}{\overline{x}_C + \overline{X} - 2S_x} \right]$	-1	S_x	-1
$T_{14} = \left[\kappa_1 \overline{y}_C + \kappa_2 (N\overline{x}_C - \rho_x) + (1 - \kappa_1 - \kappa_2)(N\overline{X} - \rho_x) \right] \exp\left[\dfrac{N(\overline{x}_C - \overline{X})}{N(\overline{x}_C + \overline{X}) - 2\rho_x} \right]$	N	ρ_x	-1

Table 2.1: Some Members of Classes of Estimator T_1

Estimators	Constants			
T_2	μ	ν	β	γ
$T_{21} = \overline{y}_C\left[\kappa_1 + \kappa_2\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\}\right]\exp\left[\dfrac{\overline{x}_C - \overline{X}}{2 + \overline{x}_C + \overline{X}}\right]$	1	1	1	-1
$T_{22} = \overline{y}_C\left[\kappa_1 + \kappa_2\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\}\right]\exp\left[\dfrac{\rho_x(\overline{x}_C - \overline{X})}{2 + \rho_x(\overline{x}_C + \overline{X})}\right]$	ρ_x	1	1	-1
$T_{23} = \overline{y}_C\left[\kappa_1 + \kappa_2\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\}\right]\exp\left[\dfrac{(\overline{x}_C - \overline{X})}{(\overline{x}_C + \overline{X}) - 2\rho_x}\right]$	-1	ρ_x	1	-1
$T_{24} = \overline{y}_C\left[\kappa_1 + \kappa_2\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\}\right]\exp\left[\dfrac{(\overline{x}_C - \overline{X})}{(\overline{x}_C + \overline{X}) + 2N}\right]$	1	N	1	-1

Table 2.2: Some Members of Classes of Estimator T_2

Estimators	Constants		
T_3	η_1	λ_1	β_1
$T_{31} = \overline{y}_{CSS}\left[\kappa_1\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\} + \kappa_2\,\exp\left\{\dfrac{(\overline{X} - \overline{x}_C)}{2 + (\overline{X} + \overline{x}_C)}\right\}\right]$	1	1	1
$T_{32} = \overline{y}_{CSS}\left[\kappa_1\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\} + \kappa_2\,\exp\left\{\dfrac{(\overline{X} - \overline{x}_C)}{(\overline{X} + \overline{x}_C) + 2N}\right\}\right]$	-1	-N	1
$T_{33} = \overline{y}_{CSS}\left[\kappa_1\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\} + \kappa_2\,\exp\left\{\dfrac{(\overline{X} - \overline{x}_C)}{(\overline{X} + \overline{x}_C) - 2S_x}\right\}\right]$	1	$-S_x$	1
$T_{34} = \overline{y}_{CSS}\left[\kappa_1\left\{2 - \dfrac{\overline{x}_C}{\overline{X}}\right\} + \kappa_2\,\exp\left\{\dfrac{(\overline{X} - \overline{x}_C)}{(\overline{X} + \overline{x}_C) + 2S_x}\right\}\right]$	1	S_x	1

Table 2.3: Some Members of Classes of Estimator T_3

Note: Here lowercase 'C' stands for CSS (circular systematic sampling)

4.5 Efficiency comparison

In order to compare the performance of the proposed classes of estimators based on CSS with the estimators based on SRSWOR, we can use the following estimators ($\overline{y}, \overline{y}_{lr}, \overline{y}'_{lr}$) in SRSWOR

$$V\left(\overline{y}\right) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2$$

$$AV\left(\overline{y}_{lr}\right) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2[1 - \rho_{yx}^2]$$

and

$$AV\left(\overline{y}_{lr}\right) = \left(\frac{1}{n'} - \frac{1}{N}\right)S_y^2 + \left(\frac{1}{n} - \frac{1}{n'}\right)S_y^2[1 - \rho_{yx}^2]$$

Note: Here using SRSWOR design, at first phase a large sample s′ of size n′ (n′< N) is selected randomly to estimate $\overline{X}$ only. In second phase, a sub-sample s of size n form n′ units is drawn randomly to estimate $\overline{Y}$ where n′=m n.

It is not easy to make analytical comparison of the proposed classes of estimators. Now we can get numerical results of variance and minimum AV of the considered estimators in CSS along with the variance and minimum AV of the estimators in SRSWOR. For this purpose, we use population data set as earlier considered by Koyuncu and Kadilar (2009) and Singh and Solanki (2013). The data concerns primary and secondary schools of 923 districts of Turkey in 2007. The description of variables is given below

y = number of teachers in both primary and secondary school;
x = number of students in both primary and secondary school.

N =923   n'=360   n =180   m =2 $\overline{X}$ =11440.5 $\overline{Y}$ =436.43

S_y =749.94 S_x =21331.13 ρ_{yx}=0.9543 ρ_y =-0.00255 ρ_x =-0.00316

For two-phases, one can select $1 < m < 5$ (as we mentioned earlier $1 < m < k$). All possible values of m are considered and numerical results are provided only for $m = 2$. Because it is observed that for $m = 2$, all the considered estimators are more efficient in CSS than SRSWOR. For $m = 3$ and $m= 4$, the estimators under SRSWOR perform a little better than CSS. So in this numerical example $m = 2$ can be the best choice among others.

Following is the description of the considered estimators in Table 3. The estimators $(\overline{y}, \overline{y}_{lr}, t_\bullet)$ are based on SRSWOR and $(\overline{y}_{CSS}, \overline{y}'_{lr}, T_\bullet)$ on CSS, further these estimators are also considered for two phase sampling.

Estimators	V/AV under single phase		V/ AV under Two phase	
	CSS	**SRSWOR**	**CSS**	**SRSWOR**
$\overline{y}$	2515.17	1698.75	2515.17	1698.75
$\overline{y}_{lr}$	224.62	151.71	1092.44	925.23
T_{11}	169.71	135.90	1076.78	**919.80**
T_{12}	188.12	141.38	1079.56	921.75
T_{13}	153.95	**131.49**	1087.49	922.96

T_{14}	218.17	149.58	1089.95	923.91
T_{21}	110.46	108.967	1074.097	922.43
T_{22}	108.012	**107.1573**	1072.78	922.26
T_{23}	110.44	108.96	1089.814	**921.82**
T_{24}	123.63	113.37	1077.198	922.82
T_{31}	206.54	141.184	1092.302	**923.86**
T_{32}	205.57	141.176	1092.23	923.863
T_{33}	203.69	141.134	1092.08	923.87
T_{34}	201.826	**141.0869**	1091.90	923.87

Table 3: Minimum Asymptotic variance of considered estimators under single and two phase sampling

In Table 3, it can be seen that the variance of $\overline{y}_{CSS}$ is smaller than the variance of $\overline{y}$. Also, the asymptotic variance of considered estimator under CSS is smaller then under SRSWOR. Hence, we conclude that the estimators based on CSS are more efficient than the estimator based on SRSWOR. Note that ρ_y and ρ_x both are less than $-\dfrac{1}{(N-1)}$. It is also observed that all the considered estimators $T_{1(.)}, T_{2(.)}, T_{3(.)}$ are more efficient than the regression estimator $\overline{y}_{lr}$. Furthermore, in first phase sampling the estimator T_{13} in class T_1, T_{22} in class T_2 and T_{34} in class T_3 provides minimum asymptotic variance among others but in two phase sampling T_{11} in class T_1, T_{23} in class T_2 and T_{31} in class T_3 provides efficient result. Henceforth, the estimator T_{22} (in 1st phase) and T_{11} (in two phase), results the best one in terms of efficiency among all considered estimators. Hence, from Table 3, we can conclude that the class T_2 in case of single-phase and T_1 for two-phase may be the best choice among others.

5. Non-response problem under CSS

When a sample of size n is selected from N circular systematic samples to collect information of Y, then incomplete or missing information might be present. The reasons non-response problem occurrence may vary in different situations. For instance, the reasons of non-response in the data set considered in previous the section may be due to strikes, holidays etc.

When non-response occurs in a CSS, we can follow the well-known Hansen and Hurwitz (1946) non-respondents sub-sampling technique. Suppose that n_1 units out of *n* can supply information on *Y* and remaining $n_2 = n - n_1$ units are taken as non-respondents. Following the technique of Hansen and Hurwitz (1946), a sub sample of size $n_r = \dfrac{n_2}{l}, (l > 1)$ is selected by SRSWOR from n_2 non-respondent units. Assume that all n_r units show full response on second call (of course *nr* must be an integer and if it isn't so, it is necessary to round). The population is said to be divided into two groups U_1 and U_2 of sizes N_1 and N_2, where U_1 is a group of respondents that would give response on the first call and U_2 is non-respondents group which could respond on the second call. Obviously N_1 and N_2 are unknown quantities.

The estimator proposed by Hansen and Hurwitz (1946) is given by-
$$\overline{y}^O = d_1 \overline{y}_1 + d_2 \overline{y}_{2r} \tag{22}$$
where
$$\overline{y}_1 = \frac{\sum_{j=1}^{n_1} y_{ij}}{n_1}, \quad \overline{y}_{2r} = \frac{\sum_{j=1}^{n_1} y_{ij}}{n_r}, \quad d_1 = \frac{n_1}{n} \text{ and } d_2 = \frac{n_2}{n}$$

The estimator $\overline{y}^*$ is unbiased and has variance
$$V(\overline{y}_{CSS}^O) = \widetilde{S}_y^2 + wS_{y(2)}^2 = \widetilde{\widetilde{S}}_y^2 \tag{23}$$
where
$$\widetilde{\widetilde{C}}_y^2 = \frac{\widetilde{\widetilde{S}}_y^2}{\overline{Y}^2}, \quad w = \frac{N_2(l-1)}{nN}, \quad \overline{Y}_2 = \frac{\sum_i^{n_2} y_{ij}}{N_2} \text{ and } S_{y(2)}^2 = \frac{\sum_i^{N_2}\sum_j^{n_2}(y_{ij} - \overline{Y}_2)^2}{n_2(N_2 - 1)}$$

The linear regression estimator defined in (1), in case of non-response in Y, can be written as
$$\overline{y}_{lr}^O(c) = \overline{y}_{CSS}^O + \hat{\beta}_{yx}(\overline{X} - \overline{x}_{CSS}) \tag{24}$$
The asymptotic variance of $\overline{y}_{lr}^O(c)$ is given by
$$AV(\overline{y}_{lr}^O(c)) = \widetilde{S}_y^2(1 - \widetilde{\rho}_{yx}^2) + wS_{y(2)}^2 \tag{25}$$
When there is a non response in Y and $\overline{X}$ is unknown, then (3) becomes
$$\overline{y}'^O_{lr}(c) = \overline{y}_{CSS}^* + \hat{\beta}_{yx}(\overline{x}'_{CSS} - \overline{x}_{CSS}) \tag{26}$$
The asymptotic variance of $\overline{y}'^*_{lr}(c)$ is given by
$$AV(\overline{y}'_{lr}(c)) = \widetilde{S}_y^2(1 - \widetilde{\widetilde{\rho}}_{yx}^2) + wS_{y(2)}^2$$

where, $\widetilde{\widetilde{\rho}}_{yx}^2 = \dfrac{\widetilde{\widetilde{S}}_{yx}^2}{\widetilde{S}_y^2 \widetilde{\widetilde{S}}_x^2}$ are such that

5.1 Suggested estimators under non-response

Now the suggested classes ((T_1, T_2, T_3)) in presence of non response in Y, could be expressed as

$$T_1^O = \left[\kappa_1 \overline{y}_{CSS}^O + \kappa_2 \overline{x}_{CSS}^* + (1 - \kappa_1 - \kappa_2)\overline{X}^*\right] \exp\left[\alpha \frac{\overline{X}^* - \overline{x}_{CSS}^*}{\overline{X}^* + \overline{x}_{CSS}^*}\right] \tag{27}$$

$$T_2^O = \overline{y}_{CSS}^O \left[\kappa_1 + \kappa_2 \left\{2 - \frac{\overline{x}_{CSS}}{\overline{X}}\right\}^\beta\right] \exp\left[\gamma \frac{(\mu\overline{X} + \nu) - (\mu\overline{x}_{CSS} + \nu)}{(\mu\overline{X} + \nu) + (\mu\overline{x}_{CSS} + \nu)}\right] \tag{28}$$

and

$$T_3^O = \overline{y}_{CSS}^O \left[\kappa_1 \left\{2 - \frac{\overline{x}_{CSS}}{\overline{X}}\right\}^{\beta_1} + \kappa_2 \exp\left\{\frac{\eta_1(\overline{X} - \overline{x}_{CSS})}{(\overline{X} + \overline{x}_{CSS}) + 2\lambda_1}\right\}\right] \tag{29}$$

The asymptotic bias of (T_1^O, T_2^O, T_3^O) will be same of (T_1, T_2, T_3). The minimum asymptotic variance of (T_1^O, T_2^O, T_3^O) under non response is respectively given by

$$AV\left(T_2^O\right) = \left[N + \kappa_1^2 N_{11} + \kappa_2^2 N_{12} - 2\kappa_1 N_{13} - 2\kappa_2 N_{14} + 2\kappa_1 \kappa_2 N_{15}\right] \tag{30}$$

Minimisation of $AV(T_1)$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_1^* = \left(\frac{N_{12}N_{13} - N_{14}N_{15}}{N_{11}N_{12} - N_{15}^2}\right) \text{ and } \kappa_2^* = \left(\frac{N_{11}N_{14} - N_{13}N_{15}}{N_{11}N_{12} - N_{15}^2}\right)$$

where

$$N = \phi^2 + \frac{\alpha^2 \tau_1^2 \overline{X}^{*2} \widetilde{C}_x^2}{2} + 2\phi\phi_1 \widetilde{C}_x^2,$$

$$N_{11} = \overline{Y}^2 \widetilde{\widetilde{C}}_y^2 + \phi^2 + \frac{\alpha^2 \tau_1^2 \phi^2}{4}\widetilde{C}_x^2 - 2\phi\phi_2 \widetilde{C}_x^2 + 2\alpha\tau_1 \phi \overline{Y}\widetilde{C}_{yx}$$

$$N_{12} = \eta^2 \overline{X}^2 \widetilde{C}_x^2,$$

$$N_{13} = \phi^2 + \frac{\alpha\tau_1 \overline{Y}(\phi + \overline{X}^*)}{2}\widetilde{C}_{yx} + \frac{\alpha^2 \tau_1^2 \overline{X}^* \phi}{2}\widetilde{C}_x^2 + \phi\widetilde{C}_x^2(\phi_1 - \phi_2)$$

$$N_{14} = \eta\alpha\tau_1 \overline{X}(\phi + \overline{X}^*)\widetilde{C}_x^2/2, \; N_{15} = \eta\alpha\tau_1 \phi\overline{X}\widetilde{C}_x^2 + \eta\overline{X}\overline{Y}\widetilde{C}_{yx}$$

$$AV\left(T_2^O\right) = \bar{Y}^2\left[\kappa_1^2 N_{21} + \kappa_2^2 N_{22} - 2\kappa_1 N_{23} - 2\kappa_2 N_{24} + 2\kappa_1\kappa_2 N_{25}\right] \qquad (31)$$

Minimisation of $AV\left(T_2\right)$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_1^* = \left(\frac{N_{22}N_{23} - N_{24}N_{25}}{N_{21}N_{22} - N_{25}^2}\right) \text{ and } \kappa_2^* = \left(\frac{N_{21}N_{24} - N_{23}N_{25}}{N_{21}N_{22} - N_{25}^2}\right)$$

where

$$N_{21} = \left[1 + \widetilde{\widetilde{C}}_y^2 + \left\{\frac{\gamma^2\tau_2^2}{4} + 2\phi_3\right\}\widetilde{C}_x^2 - 2\gamma\tau_2\widetilde{C}_{yx}\right],$$

$$N_{22} = \left[1 + \widetilde{\widetilde{C}}_y^2 + \left\{\phi_5 + 2\phi_4\right\}\widetilde{C}_x^2 - 4\phi_5\widetilde{C}_{yx}\right],$$

$$N_{23} = \left[1 + \phi_3\widetilde{C}_x^2 - \frac{\gamma\tau_2\widetilde{C}_{yx}}{2}\right], \quad N_{24} = \left[1 + \phi_4\widetilde{C}_x^2 - \phi_5\widetilde{C}_{yx}\right]$$

$$N_{25} = \left[1 + \widetilde{\widetilde{C}}_y^2 + \phi_4\widetilde{C}_x^2 - 2\phi_5\widetilde{C}_{yx} - \beta\tau_2\widetilde{C}_{yx} + \frac{\gamma\tau_2\phi_5\widetilde{C}_x^2}{2} + \phi_3\widetilde{C}_x^2\right]$$

$$AV\left(T_3^O\right) = \bar{Y}^2\left[\kappa_1^2 N_{31} + \kappa_2^2 N_{32} - 2\kappa_1 N_{33} - 2\kappa_2 N_{34} + 2\kappa_1\kappa_2 N_{35}\right] \qquad (32)$$

Minimisation of $AV\left(T_3\right)$ is achieved for the optimum choice of constants κ_1 and κ_2

$$\kappa_1^* = \left(\frac{N_{32}N_{33} - N_{34}N_{35}}{N_{31}N_{32} - N_{35}^2}\right) \text{ and } \kappa_2^* = \left(\frac{N_{31}N_{34} - N_{33}N_{35}}{N_{31}N_{32} - N_{35}^2}\right)$$

where

$$N_{31} = \left[1 + \widetilde{\widetilde{C}}_y^2 + \beta_1\widetilde{C}_x^2 - 4\beta_1\widetilde{C}_{yx}\right], N_{32} = \left[1 + \widetilde{\widetilde{C}}_y^2 + \tau_3^2\widetilde{C}_x^2 - 2\tau_3\widetilde{C}_{yx}\right],$$

$$N_{33} = \left[1 - \frac{\beta_1(\beta_1 - 1)}{2}\widetilde{C}_x^2 - \beta_1\widetilde{C}_{yx}\right], \quad N_{34} = \left[1 + \frac{3\tau_3^2}{8}\widetilde{C}_x^2 - \frac{\tau_3}{2}\widetilde{C}_{yx}\right]$$

$$N_{25} = \left[1 + \widetilde{\widetilde{C}}_y^2 + \phi_6\widetilde{C}_x^2 - 2\phi_7\widetilde{C}_{yx}\right]$$

When $\overline{X}$ is unknown, the biases of the classes $\left(T_1^{\prime O}, T_2^{\prime O}, T_3^{\prime O}\right)$ will be same of $\left(T_1^{\prime}, T_2^{\prime}, T_3^{\prime}\right)$. For the minimum asymptotic variances of these classes, only replacing the terms $\left(\widetilde{C}_x^2, \widetilde{C}_{yx}\right)$ by and $\left(\widetilde{\widetilde{C}}_x^2, \widetilde{\widetilde{C}}_{yx}\right)$.

5.2 Efficiency comparison

To compare the performance of the suggested estimators based on CSS with the estimators based on SRSWOR in the presence of non response in Y, we can use the following estimators ($\bar{y}^O, \bar{y}_{lr}^O, \bar{y}_{lr}^{\prime O}$) in SRSWOR

$$V(\overline{y}^O) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2 + wS_{y(2)}^2$$

$$AV(\overline{y}_{lr}^O) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2[1 - \rho_{yx}^2] + +wS_{y(2)}^2$$

and

$$AV(\overline{y}_{lr}'^O) = \left(\frac{1}{n'} - \frac{1}{N}\right)S_y^2 + \left(\frac{1}{n} - \frac{1}{n'}\right)S_y^2[1 - \rho_{yx}^2] + +wS_{y(2)}^2$$

We take all the possibilities for weights of the missing values (10%, 20%, 30%, 40%) etc and observe that the relative efficiency of the considered estimators is not affected by different weights of missing values. Although numerical results are different for different weights, the behaviour of results is similar in all cases. Hence, numerical results are provided only for 10% weight of missing values and consider last 92 values as non-respondents.

$$\overline{Y}_2 = 522.80, \ S_{y(2)} = 876.42, \ N_2 = 92, \ l = 2$$

Due to presence of non-response, extra variability is introduced in estimators. As expected, the variability of all considered estimators with incomplete information (see Table 4) is higher than the estimators with complete response (Table 4). Moreover, as expected, for $l > 2$, the asymptotic variance of the estimators become higher, that's why we are taking only the case when $l=2$.

Following is the description of the considered estimators in Table 4. The estimators $(\overline{y}, \overline{y}_{lr}, t_\bullet^O)$ are based on SRSWOR and $(\overline{y}_{CSS}, \overline{y}_{lr}', T_\bullet^O)$ on CSS, further these estimators are also considered for two phase sampling.

Estimators	V/AV under single phase		V/ AV under Two phase	
	CSS	**SRSWOR**	**CSS**	**SRSWOR**
$\overline{y}^O$	2940.51	2124.09	2940.51	2124.09
$\overline{y}_{lr}^O$	649.96	577.05	1517.78	1350.57
T_{11}^O	595.385	**562.17**	1496.95	**1346.33**
T_{12}^O	614.900	568.25	1506.421	1348.60
T_{13}^O	639.042	574.60	1513.933	1349.59
T_{14}^O	644.75	576.33	1516.70	1350.23
T_{21}^O	558.138	547.83	1508.41	1344.73
T_{22}^O	553.233	**546.31**	1507.41	**1344.53**
T_{23}^O	558.123	547.83	1508.414	1350.65
T_{24}^O	569.94	551.53	1510.747	1350.51
T_{31}^O	640.865	573.34	1518.804	**1348.88**

T_{32}^O	640.098	573.32	1518.895	1348.90
T_{33}^O	638.609	573.28	1519.028	1348.93
T_{34}^O	637.1094	**573.24**	1519.115	1348.95

Table 4: The minimum V/AV of the considered estimators

From Table 4 we can see that ($T_{1\bullet}^O, T_{2\bullet}^O, T_{3\bullet}^O$) are more efficient than the regression estimator $\overline{y}_{lr}^O$. Furthermore, among all the classes of estimators T_{22}^O is most efficient (both in single and two phase sampling).

6. Conclusion

In the present study, some improved estimators are proposed and their properties are studied under circular systematic sampling. The problem is also extended to the case of non response in CSS. From Tables 3 and 4, it has been observed that the proposed estimators are more efficient in CSS than SRSWOR for single-phase and two-phase sampling.

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Appendix

In this section, we have given the complete procedure for obtaining the asymptotic bias and asymptotic variance of the suggested estimators (T_1, T_2, T_3). From equation (6), T_1 in terms of δ_i's, could be expressed as

$$T_1 = \left[\kappa_1 \overline{Y}(1+\delta_y) + \kappa_2\{\eta(1+\delta_x)\overline{X}+\lambda\} + +(1-\kappa_1-\kappa_2)\overline{X}^*\right]\exp\left[\alpha\frac{\overline{X}^* - \{\eta(1+\delta_x)\overline{X}+\lambda\}}{\overline{X}^* + \{\eta(1+\delta_x)\overline{X}+\lambda\}}\right]$$

(33)

On retaining only the terms up to the second degree of δ_i's, we have

$$T_1 - \overline{Y} = \left[\phi - \frac{\alpha\tau\overline{X}^*}{2}\delta_x + \phi_1\delta_x^2 + \kappa_1\left\{\delta_y\overline{Y} - \phi + \frac{\alpha\tau\phi}{2}\delta_x + \phi_2\delta_x^2\right\}\right] \quad (34)$$

After taking expectation both sides of equation (34), we get the asymptotic bias of T_1 as

$$AB(T_1) = \left[\phi + \phi_1\widetilde{C}_x^2 + \kappa_1\left\{A_2\phi_2 - A - \frac{\overline{Y}\alpha\tau_1\widetilde{C}_{yx}}{2}\right\} - \kappa_2\frac{\eta\overline{X}\alpha\tau_1\widetilde{C}_x^2}{2}\right]$$

Squaring and taking expectations of both sides of equation (34), we have the required asymptotic variance of T_1, respectively given by equation (9).

Now from equation (14), T_2 in terms of δ_i's, could be expressed as

$$T_2 = (1+\delta_y)\overline{Y}\left[\kappa_1 + \kappa_2\left\{2 - \frac{(1+\delta_x)\overline{X}}{\overline{X}}\right\}\right]\exp\left[\gamma\frac{(\mu\overline{X}+\nu) - (\mu(1+\delta_x)\overline{X}+\nu)}{(\mu\overline{X}+\nu) + (\mu(1+\delta_x)\overline{X}+\nu)}\right]$$

(35)

After expanding equation (35), we have

$$T_2 = \overline{Y}\left[k_1\left\{1+\delta_y - \frac{\gamma\tau_2\delta_x}{2} + \phi_3\delta_x^2 - \frac{\gamma\tau_2\delta_x\delta_y}{2}\right\} + k_2\left\{1+\delta_y - \phi_3\delta_x + \phi_4\delta_x^2 - \phi_5\delta_x\delta_y\right\}\right]$$

(36)

Subtracting $\overline{Y}$ and taking expectations both sides of above equation (36), we get the requires bias of T_2 as

$$AB(T_2) = \overline{Y}\left[\kappa_1\left\{1 + \phi_3\widetilde{C}_x^2 - \frac{\gamma\tau_2\widetilde{C}_{yx}}{2}\right\} + \kappa_2\left\{1 + \phi_4\widetilde{C}_x^2 - \phi_5\widetilde{C}_{yx}\right\} - 1\right]$$

Subtracting and squaring and than taking expectations of both sides of equation (36), we have the required asymptotic variance of T_2, respectively given by equation (16).

Now from equation (18), T_3 in terms of δ_i's, could be expressed as

$$T_3 = \left(1+\delta_y\right)\overline{Y}\left[k_1\left\{2-(1-\delta_x)^{\beta_1}\right\} + \kappa_2\exp\left\{\frac{-\eta_1\overline{X}\delta_x}{2(\eta_1\overline{X}+\lambda_1)+\eta_1\overline{X}\delta_x}\right\}\right]$$

(37)

After expanding equation (37), we have

$$T_3 = \overline{Y}\left[k_1\left\{1+\delta_y - \beta_1\delta_x - \frac{\beta_1(\beta_1-1)}{2}\delta_x^2 - \beta_1\delta_x\delta_y\right\} + k_2\left\{1+\delta_y - \frac{\tau_3\delta_x}{2} + \frac{3\tau_3^2\delta_x^2}{8} - \frac{\tau_3}{2}\delta_x\delta_y\right\}\right]$$

(38)

Subtracting $\overline{Y}$ and taking expectations both sides of above equation (38), we get the requires bias of T_3 as

$$AB(T_3) = \overline{Y}\left[\kappa_1\left\{1 + \frac{\beta_1(\beta_1-1)}{2}\widetilde{C}_x^2 - \beta_1\widetilde{C}_{yx}\right\} + \kappa_2\left\{1 + \frac{3\tau_3^2}{8}\widetilde{C}_x^2 - \frac{\tau_3}{2}\widetilde{C}_{yx}\right\} - 1\right]$$

Subtracting and squaring and than taking expectations of both sides of equation (38), we have the required asymptotic variance of T_3, respectively given by equation (20).

Note: The optimum values of κ_1 and κ_2 can be obtained by differentiating their respective AV (T) partially w.r.t. to κ_1 and κ_2 an equate it to zero as

$$\frac{\partial AV(T)}{\partial\kappa_1} = 0 \text{ and } \frac{\partial AV(T)}{\partial\kappa_2} = 0$$

(39)

After solving (39) for κ_1 and κ_2, we get the required optimum values for κ_1 & κ_2.