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DISCRETIZATION OF BURR-TYPE III DISTRIBUTION

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Abstract

In this paper we propose a discrete analogue of Burr-type III distribution using a general approach of Discretizing a continuous distribution. It may be worth exploring the possibility of developing a discrete version of two parameter Burr-type III distribution, so that same can be used for modeling a discrete data. Discrete Burr-type III distribution is suggested as a suitable reliability model to fit a range of discrete life time data, as it is shown that hazard rate function can attain monotonic increasing (decreasing) shape for certain values of parameters. The equivalence of discrete Burr-type III (DBD-III) and continuous Burr-type III (BD-III) distributions has been established. Various theorems relating Burr Type III distribution with other statistical distributions have also been proved.

Key Words: Burr-type III distribution, discrete lifetime models, reliability, failure rate.

1. Introduction

In reliability theory a plethora of continuous life models is now available in the subject to portray the survival behavior of a component or a system. Many continuous life distributions have been studied in details (see for example Kapur and Lamberson (1997), Lawless (1982) and Sinha (1986). However, it is sometimes impossible or inconvenient in life testing experiments to measure the life length of a device on a continuous scale. E.g. the lifetime of an on/off switching device is a discrete random variable, or life length of a device receiving a number of shocks it sustain before it fails is also a discrete random variable. In the recent past special roles of discrete distribution are getting recognition from survival analysts. Many continuous distributions have been discretised, e.g. the Geometric and Negative binomial distributions are the discrete versions of Exponential and Gamma distributions. Nakagawa (1975) discretised the Weibull distribution. The discrete versions of the normal and rayleigh distributions were also proposed by Dilip Roy (2003,2004). Discrete analogues of maxwell, two parameter Burr XII and Pareto distributions were also proposed by Krishna and punder (2007,2009). Recently inverse Weibull distribution were also discretised by Mansour Aghababaei Jazi, Chin-Diew lai and Mohammad Hussein Alamatsaz (2010).

The present paper deals with the problem of discretization of Burr-type III (BD-III) distribution, as there is a need to find more plausible discrete life time distributions to fit to various life time data.

2. Discretising a continuous distribution

Roy (1993) pointed out that the univariate geometric distribution can be viewed as a discrete concentration of a corresponding exponential distribution in the following manner:

$p\,[X = x] = s(x) - \, s\,(x + 1)$ When x = 0, 1, 2,…..

Where X is discrete random variable following geometric distribution with probability mass functions as

$p\,(x) = \,\theta^x(1 - \theta)$ x = 0,1,2,…….

Where s(x) represents the survival function of an exponential distribution of the form s(x) = exp(-λx) clearly

θ = exp(-λ), 0 < θ < 1.

Thus, one to one correspondence between the geometric distribution and the exponential distribution can be established, the survival functions being of the same form.

The general approach of dicretizing a continuous variable is to introduce a greatest integer function of X i.e., [X] (the greatest integer less than or equal to X till it reaches the integer), in order to introduce grouping on a time axis.

If the underlying continuous failure time X has the survival function s(x) = p(X $\geq$ x) and times are grouped into unit intervals, so that the discrete observed variable is dX = [X].

The probability mass function of dX can be written as

$$p(x) = p(dX = x) = p(x \leq X < x + 1) = \emptyset(x + 1) - \emptyset(x)$$
$$= s(x) - \, s(x + 1), \quad x = 0,1,2……$$

$\emptyset(x)$ being the cumulative distribution function of rv X.

In reliability theory, many classification properties and measures are directly related to the functional form of the survival function. The increasing failure rate (IFR), decreasing failure rate (DFR), Increasing failure rate average (IFRA), decreasing failure rate average (DFRA), new better than used (NBU), new worse than used (NWU), new better (worse) than used in expectation NBUE (NWUE) and increasing (decreasing) mean residual lifetime IMRL (DMRL) etc. are examples of such class properties as may be seen from Barlow and Proschan (1975). If discretization of a continuous life distribution can retain the same functional form of the survival function then many reliability measures and class properties will remain unchanged. In this sense, we consider the discrete concentration concept as a simple approach that can generate a discrete life distribution model.

Thus given any continuous life variable with survival function s(x) we define a discrete lifetime variables x with probability mass function p(x) given by

$p(x) = s(x) - \, s(x + 1)$ x = 0, 1, 2….

We would like to make use of this concept for the purpose of discretizing Burr-type III distribution. A random variable *X* is said to follow Burr-type III distribution with parameter *(c, k)* if its probability density function is given by

$f(x) = \frac{ck}{x^{c+1}(1+x^{-c})^{k+1}}$ *x > 0; c > 0; k >0*

2.1The various reliability measures of a random variable X are given by

(a) Survival function

$s(x) = 1 - \int_0^x f(x)dx$

$$= 1 - \int_0^x \frac{ck}{x^{c+1}(1+x^{-c})^{k+1}} dx$$
$$= 1 - (1 + x^{-c})^{-k} \qquad x > 0;\ c > 0;\ k > 0$$

(b) The failure rate is given by

$$r(x) = \frac{ck}{[x^{c+1}(1+x^{-c})^{k+1}]\,[1-(1+x^{-x})^{-k}]}$$

$x > 0;\ c > 0;\ k > 0$

(c) The second rate of failure is given by

$$SRF(x) = \log\left(\frac{s(x)}{s(x+1)}\right) = \log\left(\frac{1-(1+x^{-c})^{-k}}{1-(1+(x+1)^{-c})^{-k}}\right)$$

$x > 0;\ c > 0;\ k > 0$

Note that for second rate of failure

$$SRF(0) = SRF(1) \Rightarrow c = -\left[\frac{\left(2^{-k}(2-2^{-k})\right)^{-1/k} - 1}{log2}\right] = \alpha\ (say)$$

It could be seen that *SRF(x)* is decreasing in x if $c < \alpha$ and for $c > \alpha$, *SRF(0) < SRF(1)* and for all other values $x \geq 1$, *SRF(x)* decreases for all.

(d) The rth moment is

$$E(x^r) = \int_0^\infty x^r f(x) dx$$

$$= K\beta\left(1 - \frac{r}{c}, k + \frac{r}{c}\right) \text{ Where } \beta(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} dx \ , \ x > 0;\ c > 0;\ k > 0$$

3. Discrete Burr-type III distribution

A discrete Burr- type III variable, *dX* can be viewed as the discrete concentration of the continuous Burr- type III variable *X*, where the corresponding probability mass function of *dX* can be written as:

$$P(dX = x) = p(x) = s(x) - s(x+1)$$

The probability mass function takes the form

$$P(x) = \begin{cases} \theta^{log2} & x=0 \\ \theta^{\log(1+(x+1)^{-c})} - \theta^{\log(1+(x)^{-c})} & x = 1,2,3 \dots. \end{cases} \qquad 3.1$$

Where $\theta = e^{-k}$; 0<θ<1; k>0

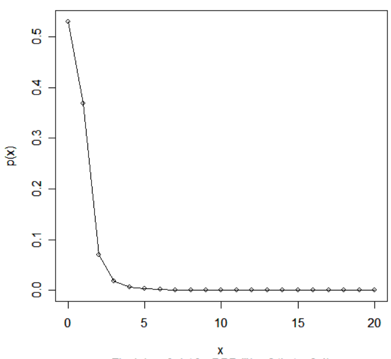

Fig.1.1 pmf plot for DBD-III(c=3,theta=0.4)

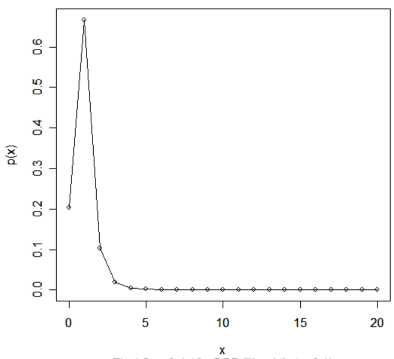

Fig.1.2 pmf plot for DBD-III(c=4,theta=0.1)

Fig.1.1 and fig.1.2 gives the pmf plot of (3.1) for (c=3, θ=0.4),(c=4, θ=0.1) respectively. The scale parameters θ completely determines the *pmf (3.1)* at *x = 0*. It should be also noted that the *p(x)* is always monotonic decreasing for *x = 1,2,3,4,….*

When $log\theta \geq \frac{log2}{\log(1+2^{-c})-log2}$

P(0) < P(1) and then *p(x)* decreases ∀ *x = 1, 2,3,…* i.e., *p(x)* is a unimodal (with mode at 1). The shape parameter c has more influence on the *pmf* than θ after x = *0,* also as the c becomes smaller, the tail of the *pmf* becomes longer.

4. Reliability measures of discrete Burr-type III random variable dX are given by

(e) Survival function

$$s(x) = p(dX \geq x) = 1 - \theta^{\log(1+x^{-c})} \qquad x = 0,1,2,3 \ldots.$$
$$c > 0; 0 < \theta < 1$$

$s(x)$ is same for continuous Burr-type III distribution and discrete Burr-type III distribution at the integer points of *x*.

(f) Rate of failure, r(x) is given by

$$r(x) = \frac{p(x)}{s(x)} = \frac{\theta^{\log(1+(x+1)^{-c})} - \theta^{\log(1+x^{-c})}}{1-\theta^{\log(1+x^{-c})}} \qquad x = 0,1,2,3 \ldots.$$
$$c > 0; 0 < \theta < 1$$

(g) Second rate of failure is given by

$$SRF(x) = log\left(\frac{1-\theta^{\log(1+x^{-c})}}{1-\theta^{\log(1+(x+1)^{-c})}}\right) \qquad x = 0,1,2,3 \ldots$$
$$c > 0; 0 < \theta < 1$$

It could be seen that *r(x)* and *SRF(x)* are the monotonic decreasing functions if

$$c < -log\left[2^{log_e\theta}\,(2 - 2^{log_e\theta})^{\frac{1}{log_e\theta}} - 1\right]/log2 \ = \alpha(\text{say})$$
$$k > 0\,;\ 0 < \theta < 1; \qquad k = -log_e\theta$$

Fig.1.3 and fig.1.4 illustrates the second rate of failure plot for DBD-III(1,0.05),(0.5,0.05) respectively. For *c > α; r(0) < r(1)* and *SRF(0) < SRF(1)* and for all other values of *x ≥ 1, r(x)* and SRF(x) decreases, clearly the hazard rates of continuous model and the discrete modal shows the same monotonocity.

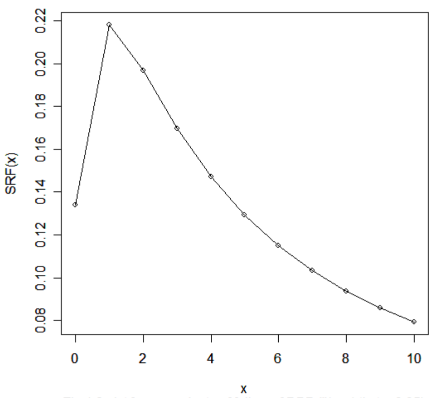

Fig.1.3 plot for second rate of failure of DBD-III(c=1,theta=0.05)

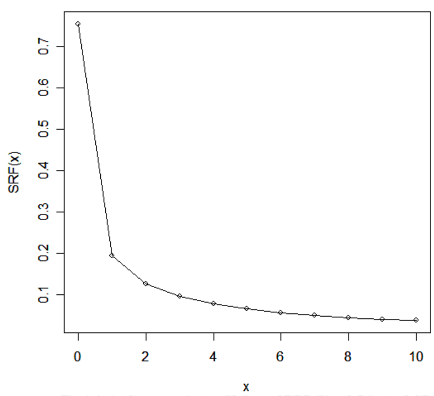

Fig.1.4 plot for second rate of failure of DBD-III(c=0.5,theta=0.05)

4.1 Moments of discrete Burr- type III distribution

$$E(x^r) = \sum_{x=0}^{\infty} x^r p(x)$$
$$= \sum_{x=1}^{\infty} [x^r - (x-1)^r]\, s(x)$$
$$= \sum_{x=1}^{\infty} [x^r - (r_{c_0} x^r (-1)^0 + r_{c_1} x^{r-1}(-1)^1 + r_{c_2} x^{r-2}(-1)^2 + r_{c_3} x^{r-3}(-1)^3 + \dots + r_{c_r} x^0 (-1)^r)]\, s(x)$$
$$\leq \sum_{x=1}^{\infty} r\, x^{r-1}\; s(x)$$
$$= \sum_{x=1}^{\infty} r\, x^{r-1} (1 - \theta^{\log(1+x^{-c})})$$
$$= \sum_{x=1}^{\infty} r \left[\frac{1}{Kx^{-ck+c-r+1+ck}} + \frac{k_{c_1}}{k_{c_2}} \frac{1}{x^{-ck+2c-r-c+1+ck}} + \dots + \frac{k_{c_{k-1}}}{k_{c_k}} \frac{1}{x^{c-r+1}}\right]$$

R.H.S expression is finite if $c > r$

Now

$$E(x) = \sum_1^{\infty} s(x)$$
$$= \sum_1^{\infty} (1 - \theta^{\log(1+x^{-c})})$$

is finite if $c > 1$

Similarly for the convergence of variance, c must be greater than 2.

5. Estimation of the parameters of discrete Burr type III distribution

Estimation of the parameters based on the ML method: Let n items be put on the test and their lifetimes are recorded as $X_1, X_2 X_3, \dots \dots X_n$. If these X_i's are assumed to be iid random variables following discrete Burr-type III distribution i.e., $\mathrm{DBD-III}(c,\theta)$, their likelihood function is given by

$$L(c,\theta;x) = \prod_{i=1}^{n} p(x_i)$$
$$= \prod_{i=1}^{n} (\theta^{\log(1+(x_i+1)^{-c})} - \theta^{\log(1+(x_i)^{-c})}) \quad (5.1)$$

And (4.8.1) can be rewritten as follows

$$L(c,\theta;x) = \prod_{i=1}^{n} \theta^{\log(1+(x_i)^{-c})} (\theta^{\varphi(x_i,c)} - 1) \quad (5.2)$$

where $\varphi(x_i, c) = \log\,[\frac{(1+(x_i+1)^{-c})}{(1+x_i^{-c})}$

Now to find the two log likelihood equations we need first to obtain the log likelihood function which is given by

$$\log L = \sum_{i=1}^{n} [\log(1 + x_i^{-c}) \log\theta + \log(\theta^{\varphi(x_i,c)} - 1)] \quad (5.3)$$

Case I: (c is known and θ is unknown)

In this case the MLE of the unknown parameter θ is $\hat{\theta}$, that is the solution of the following likelihood equation, with an observed sample this equation can be solved using an iterative numerical method.

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^{n} [\frac{\log(1+x_i^{-c})}{\hat{\theta}} + \frac{\varphi(x_i,c)\hat{\theta}^{\varphi(x_i,c)} - 1)}{\hat{\theta}^{\varphi(x_i,c)} - 1)}] = 0 \quad (5.4)$$

The solution of this equation will provide the MLE of θ by using numerical computation. The MLE's of the reliability, failure rate and the second rate of failure functions are based on the invariance property of the ML, respectively as follows

$$s(x) = 1 - \hat{\theta}^{\log(1+x^{-c})}$$

$$r(x) = \frac{\hat{\theta}^{\log(1+(x+1)^{-c})} - \hat{\theta}^{\log(1+x^{-c})}}{1-\hat{\theta}^{\log(1+x^{-c})}}$$

And $\mathrm{SRF}(x) = \log\left(\frac{1-\hat{\theta}^{\log(1+x^{-c})}}{1-\hat{\theta}^{\log(1+(x+1)^{-c})}}\right)$

Case II: (c and θ are unknown)

In this case, the solution of the following likelihood equations provide the MLE's of the unknown parameters θ and c, which are denoted by $\hat{\theta}$ and $\hat{c}$, respectively. With an observed sample these equations can be solved using an iterative numerical method. So these, the first derivative with respect to θ and c, of the log-likelihood equation (5.2) are given by

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^{n}\left[\frac{\log(1+x_i^{-c})}{\hat{\theta}} + \frac{(x_i,c)\hat{\theta}^{\varphi(x_i,c)}-1)}{\hat{\theta}^{\varphi(x_i,c)}-1)}\right] = 0$$

$$\frac{\partial \log L}{\partial c} = \sum_{i=1}^{n}\left[\frac{(-x_i^{-\hat{c}})\log\theta \log x_i}{(1+x_i^{-\hat{c}})} + \frac{\log\theta\varphi'(x_i,\hat{c})\theta^{\varphi(x_i,\hat{c})}}{\theta^{\varphi(x_i,\hat{c})}-1}\right] = 0$$

Where $\varphi'(x_i,\hat{c}) = \frac{[x_i^{-\hat{c}}\{(1+(x_i+1)^{-\hat{c}})\}\log x_i - (1+(x_i)^{-\hat{c}})\log(1+x_i)]}{\{1+(x_i+1)^{-\hat{c}}\}1+(x_i)^{-\hat{c}}}$

By using numerical computation, the solution of these normal equations will provide the MLE of θ and c. the MLE of the reliability, the failure rate and the second rate of failure functions are obtained based on the invariance property of the ML, respectively as follows

$$s(x) = 1 - \hat{\theta}^{\log(1+x^{-c})}$$

$$r(x) = \frac{\hat{\theta}^{\log(1+(x+1)^{-c})} - \hat{\theta}^{\log(1+x^{-c})}}{1-\hat{\theta}^{\log(1+x^{-c})}}$$

And $\mathrm{SRF}(x) = \log\left(\frac{1-\hat{\theta}^{\log(1+x^{-c})}}{1-\hat{\theta}^{\log(1+(x+1)^{-c})}}\right)$

6. Some theorems related to discrete Burr-type III distribution

Lemma 1: If X is a continuous rv with increasing (decreasing) failure rate IFR (DFR) distribution, then dX= [X] has a discrete increasing (decreasing) failure rate dIFR (dDFR).

Proof: (See Roy and Dasgupta, 2001)

Lemma 2: If X is a non-negative continuous rv and Y is a non-negative integer valued discrete rv, then

$$[X] \geq Y \Leftrightarrow X \geq Y$$

Proof: Note that,

$$([X] \geq Y) \subseteq (X \geq Y) \subseteq ([X] \geq [Y]) = ([X] \geq Y)$$

Where the last equality holds since Y is integer valued.

Therefore $(X \geq Y) = ([X] \geq Y)$

Theorem 1: If $X \sim BD-III(c,k)$ then *Y= [X]* $\sim DBD-III(c,\theta)$

Where $\theta = e^{-k}; 0 < \theta < 1; c > 0; k > 0$

Proof:- Consider

$$P(Y \geq y) = P[[X] \geq y]$$

$= P[X \geq y]$ By lemma 2
$= 1 - (1 + y^{-c})^{-k}$
$= 1 - \theta^{\log(1+y^{-c})}$

Which is the survival function of a discrete Burr- type III distribution i.e., *DBD-III (c, θ)*

Theorem 2: If $X \sim$ *BD-III (c, k)* then $Y = \left[[\log(1 + X^{-c})]^{-1/c}\right]$ follows discrete inverse Weibull distribution i.e., *DIW (c, θ)*

$\theta = e^{-k}$; $0 < \theta < 1$

Proof:-

$$P[Y \geq y] = P\left[\left[[\log(1 + X^{-c})]^{-1/c}\right] \geq y\right]$$
$$= P\left[[\log(1 + X^{-c})]^{-1/c} \geq y\right]$$
$$= P\left[X \geq (e^{y^{-c}} - 1)^{-1/c}\right]$$
$$= 1 - \theta^{log\left[1+\left[(e^{y^{-c}}-1)^{-1/c}\right]^{-c}\right]}$$
$$= 1 - \theta^{\log e^{y^{-c}}} = 1 - \theta^{y^{-c}}$$

Which is the survival function of a discrete inverse Weibull distribution.
Hence Y$\sim DIW(c, \theta)$

Theorem 3: If X is a non-negative *rv* and t is the positive number. Then$X_t = [X^t] \sim DBD - III\left(\frac{c}{t}, \theta\right) if\ X \sim BD - III\ (c, k)$

$\theta = e^{-k}; 0 < \theta < 1$

Proof: Let $X \sim BD - III(c, k)$ then $\forall\ x = 0,1,2, \ldots$

$$P[X_t \geq x] = P\left[[X^t] \geq x\right] = P[X^t \geq x]$$
$$= P\left[X \geq x^{1/t}\right]$$
$$= 1 - \theta^{\log(1+x^{-c/t})}$$

$\Rightarrow X_t \sim DBD - III(c/t, \theta)$

Theorem 4: If $X \sim$ BD-III (c, k), then $Y = \left[\{\log(1 + X^{-c})\}^{1/c}\right]$ follows discrete Weibull distribution i.e., DWD (c, θ) $\theta = e^{-k}$ $0 < \theta < 1$, k>0

Proof: Consider

$$P[Y \geq y] = P\left[\left[\{\log(1 + X^{-c})\}^{1/c}\right] \geq y\right]$$
$$= 1 - P\left[X \geq (e^{y^c} - 1)^{-1/c}\right]$$
$$= 1 - \left[1 - \left[(1 + \left[(e^{y^c} - 1)^{-1/c}\right]^{-c}\right]^{-k}\right]$$
$$= \theta^{y^c} \quad \text{Where } \theta = e^{-k}$$

Which is the survival function of a discrete weibull distribution

Hence $Y \sim DW(c, \theta)$

Theorem 5: Let X be random variable following continuous Burr-type III distribution with $E(X^r) < \infty$ $\forall\ r = 1,2,3 \ldots$.

Then $E(Y^r) < \infty$ where $Y = [X] \sim DB - III(c, k)$

Proof: Proof is straight forward, since $0 \leq [X] \leq X$, so clearly if
$E(X^r) < \infty$ $\forall$ r = 1,2,3……
Then $E([X]^r) < \infty$

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