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MEDIAN BASED MODIFIED RATIO ESTIMATORS WITH LINEAR COMBINATIONS OF POPULATION MEAN AND MEDIAN OF AN AUXILIARY VARIABLE

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Abstract

In this paper two new median based modified ratio estimators for the estimation of finite population mean using the linear combinations of population mean and median of the auxiliary variable have been proposed. The bias and mean squared error of the proposed estimators are derived and the mean squared errors are compared with that of the SRSWOR sample mean, ratio estimator, linear regression estimator and median based ratio estimator for certain natural populations. It is observed from the numerical comparisons that the proposed median based modified ratio estimators have outperformed the existing estimators including the linear regression estimator.

Key Words: Bias, Linear Regression Estimator, Mean Squared Error, Natural Population, Simple Random Sampling.

1. Introduction

Let $U = \{U_1, U_2, \dots, U_N\}$ be the finite population with N distinct and identifiable units. Let Y be the study variable with value Y_i measured on $U_i, i = 1, 2, 3, \dots, N$ giving a vector $Y = \{Y_1, Y_2, \dots, Y_N\}$. The problem is to estimate the population means $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ with some desirable properties like unbiasedness, minimum variance of the estimators on the basis of a random sample of size n selected from the population U. When the population parameters of the auxiliary variable X such as population mean, coefficient of variation, coefficient of kurtosis, coefficient of skewness, and median are known, a number of modified ratio estimators are proposed in the literature by extending the usual ratio estimator.

Before discussing further about the existing estimators and the proposed modified ratio estimators the notations to be used in this paper are described below:

- N - Population size
- n - Sample size
- Y - Study variable
- M - Median of the Study variable
- X - Auxiliary variable
- M_d - Median of auxiliary variable
- ρ - Correlation Co-efficient between X and Y

- $\bar{X}, \bar{Y}$ - Population means
- $\bar{x}, \bar{y}$ - Sample means
- $\bar{M}$ - Average of sample medians of Y
- m - Sample median of Y
- β - Regression coefficient of Y on X
- $B(.)$ - Bias of the estimator
- $V(.)$ - Variance of the estimator
- $MSE(.)$ - Mean squared error of the estimator
- $PRE(e,p) = \frac{MSE\ (e)}{MSE\ (p)} * 100$ – Percentage relative efficiency of the proposed estimator p with respect to the existing estimator e

The formulae for computing various measures including the variance and the covariance of the SRSWOR sample mean and sample median are as follows:

$$V(\bar{y}) = \frac{1}{^{N}C_{n}}\sum_{i=1}^{^{N}C_{n}}(\bar{y}_i - \bar{Y})^2 = \frac{1-f}{n}S_y^2\ , V(\bar{x}) = \frac{1}{^{N}C_{n}}\sum_{i=1}^{^{N}C_{n}}(\bar{x}_i - \bar{X})^2 = \frac{1-f}{n}S_x^2,\ MSE(M) = V(m) = \frac{1}{^{N}C_{n}}\sum_{i=1}^{^{N}C_{n}}(m_i - M)^2$$

$$Cov(\bar{y},\bar{x}) = \frac{1}{^{N}C_{n}}\sum_{i=1}^{^{N}C_{n}}(\bar{x}_i - \bar{X})\ (\bar{y}_i - \bar{Y}) = \frac{1-f}{n}\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})\ (Y_i - \bar{Y})\ ,$$

$$Cov(\bar{y},m) = \frac{1}{^{N}C_{n}}\sum_{i=1}^{^{N}C_{n}}(m_i - M)\ (\bar{y}_i - \bar{Y})$$

$$C'_{xx} = \frac{V(\bar{x})}{\bar{X}^2},\quad C'_{mm} = \frac{V(m)}{M^2}, C'_{ym} = \frac{Cov(\bar{y},m)}{M\bar{Y}},\quad C'_{yx} = \frac{Cov(\bar{y},\bar{x})}{\bar{X}\bar{Y}}$$

where $f = \frac{n}{N};\ S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2, S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2$

The simplest estimator of population mean $\bar{Y}$ is the sample mean $\bar{y}$ of size n obtained by using simple random sampling without replacement. The variance of SRSWOR sample mean $\hat{\bar{Y}}=\hat{\bar{Y}}_r=\bar{y}$ is given by

$$V(\hat{\bar{Y}}_r) = \frac{1-f}{n}S_y^2 \tag{1.1}$$

The sampling theory describes a wide variety of techniques for using auxiliary information to obtain more efficient estimators like ratio, product and regression estimators under certain conditions. For further details on ratio and regression estimators the readers are referred to see Cochran (1977) and Murthy (1967). Ratio estimator improves the precision of the estimate of the population mean or total of a study variable Y provided the auxiliary variable X which is positively correlated with the study variable Y. The ratio estimator for estimating the population mean $\bar{Y}$ of the study variable Y is defined as $\hat{\bar{Y}}_R = \frac{\bar{y}}{\bar{x}}\bar{X} = \hat{R}\bar{X}$. The bias and the mean squared error are as given below:

$$B\left(\hat{\bar{Y}}_R\right) = \bar{Y}\{C'_{xx} - C'_{yx}\} \tag{1.2}$$

$$MSE\left(\hat{\bar{Y}}_R\right) = V(\bar{y}) + R^2 V(\bar{x}) - 2R\,Cov(\bar{y}, \bar{x}) \quad \text{where } R = \frac{\bar{Y}}{\bar{X}} \tag{1.3}$$

The other important and the optimum estimator for estimating the population mean $\bar{Y}$ of the study variable Y using the auxiliary information is the linear regression estimator. The linear regression estimator and its variance are given below:

$$\hat{\bar{Y}}_{lr} = \bar{y} + \beta(\bar{X} - \bar{x}) \tag{1.4}$$

$$V\left(\hat{\bar{Y}}_{lr}\right) = V(\bar{y})(1 - \rho^2) \text{ where } \rho = \frac{Cov\,(\bar{y}, \bar{x})}{\sqrt{V(\bar{x}) * V(\bar{y})}} \tag{1.5}$$

For further details on the modified ratio estimators with the known population parameters of the auxiliary variable such as coefficient of variation, skewness, kurtosis, correlation coefficient, quartiles and their linear combinations, the readers are referred to see the following articles: Kadilar and Cingi (2004, 2006a, b, 2009), Koyuncu and Kadilar (2009), Rajesh Tailor, Ritesh Tailor, Rajesh Parmar and Manish Kumar (2012), Singh and Kumar (2011), Subramani (2013), Subramani and Kumarapandiyan (2012a, b, c, d, 2013) and the references cited therein.

In this paper, two new median based modified ratio estimators with the linear combinations of population mean and median of the auxiliary variable have been proposed. The proposed median based ratio estimators together with the bias and the mean squared error are given in Section 2.

2. Proposed Modified Ratio Estimators

In this section, two median based modified ratio estimators using the linear combinations of population mean and median of the auxiliary variable has been suggested. The proposed modified ratio estimators $\hat{\bar{Y}}_{SPj}$, $j = 1, 2$ together with the bias and mean squared errors are given below:

The detailed derivations of the bias and the mean squared error are given in the Appendix A.

$$\hat{\bar{Y}}_{SP\,1} = \bar{y}\left[\frac{M_d\ M + \bar{X}}{M_d\ m + \bar{X}}\right] \tag{2.1}$$

$$\hat{\bar{Y}}_{SP\,2} = \bar{y}\left[\frac{\bar{X}M + M_d}{\bar{X}m + M_d}\right] \tag{2.2}$$

$$B\left(\hat{\bar{Y}}_{SPj}\right) = \bar{Y}\left\{\theta_j^{'2}\frac{v(m)}{M^2} - \theta_j^{'}\frac{Cov\,(\bar{y}, m)}{M\bar{Y}} - \theta_j^{'}\frac{Bias\,(m)}{M}\right\}, j = 1,2 \tag{2.3}$$

$$MSE\left(\hat{\bar{Y}}_{SPj}\right) = V(\bar{y}) + R^{'2}\theta_j^{'2}V(m) - 2R^{'}\theta_j^{'}Cov(\bar{y}, m)\,, \quad j = 1,2 \tag{2.4}$$

where, $R^{'} = \frac{\bar{Y}}{M}, \quad \theta_1^{'} = \frac{M_d\ M}{M_d\ M + \bar{X}}, \quad \theta_2^{'} = \frac{\bar{X}M}{\bar{X}M + M_d}$

3. Efficiency Comparison

In this section, the conditions for which the proposed modified ratio estimators will have minimum mean squared error compared to SRSWOR sample mean, ratio estimator, linear regression estimator and median based ratio estimator for estimating the finite population mean have been derived algebraically.

3.1 Comparison with that of SRSWOR sample mean

From the expressions given in (2.4) and (1.1) we have derived the conditions (see Appendix-B) for which the proposed estimators $\hat{\hat{Y}}_{SPj}$, $j = 1, 2$ are more efficient than the SRSWOR sample mean $\hat{\hat{Y}}_r$ and are given below:

$$MSE(\hat{\hat{Y}}_{SPj}) \leq V(\hat{\hat{Y}}_r) \text{ , if } 2C'_{ym} \geq \theta'_j C'_{mm} \text{ ; } j = 1, 2 \quad (3.1)$$

3.2 Comparison with that of Ratio Estimator

From the expressions given in (2.4) and (1.3) we have derived the conditions (see Appendix-B) for which the proposed estimators $\hat{\hat{Y}}_{SPj}$, $j = 1, 2$ are more efficient than the usual ratio estimator $\hat{\hat{Y}}_R$ and are given below:

$$MSE(\hat{\hat{Y}}_{SPj}) \leq MSE(\hat{\hat{Y}}_R) \text{ , if } \theta_j'^2 C'_{mm} - C'_{xx} \leq 2\left(\theta'_j C'_{ym} - C'_{yx}\right) \text{ ; } j = 1, 2 \quad (3.2)$$

3.3. Comparison with that of Linear Regression Estimator

From the expressions given in (2.4) and (1.5) we have derived the conditions (see Appendix-B) for which the proposed estimators $\hat{\hat{Y}}_{SPj}$, $j = 1, 2$ are more efficient than the usual linear regression estimator $\hat{\hat{Y}}_{lr}$ and are given below:

$$MSE(\hat{\hat{Y}}_{SPj}) \leq V(\hat{\hat{Y}}_{lr}) \text{ , if } 2\theta'_j C'_{ym} - \theta_j'^2 C'_{mm} \geq \frac{[C'_{yx}]^2}{C'_{xx}} \text{ , } j = 1, 2 \quad (3.3)$$

4. Numerical Comparison

In section 3, the conditions are derived for which the proposed median based modified ratio estimators to be performed better than the other usual estimators like, simple random sample mean, ratio estimator, linear regression estimator and so on. However it has not been proved explicitly by algebraic expressions that the proposed estimators are better than the other estimators mentioned above. Alternatively one has to resort for numerical comparisons to determine the efficiencies of the proposed estimators.

In this view, three natural populations available in the literature are used for comparing the efficiencies of the proposed modified ratio estimators with that of the existing estimators. The population 1 and 2 are taken from Daroga Singh and Chaudhary (1986) page no.177 and the population 3 is taken from Mukhopadhyay (1988) page no.155. The parameter values and constants computed for the above populations are presented in Table 4.1, 4.2; the bias for the proposed modified ratio estimators and the existing estimators computed for the three populations discussed above are presented in the Table 4.3 whereas the mean squared errors are presented in Table 4.4.

Parameters	For sample size $n = 3$		
	Popln-1	**Popln-2**	**Popln-3**
N	34	34	20
n	3	3	3
$\bar{Y}$	856.4118	856.4118	41.5
M	767.5	767.5	40.5
$\bar{X}$	208.8824	199.4412	441.95
M_d	150	142.5	407.5
R	4.0999	4.2941	0.0939
R'	1.1158	1.1158	1.0247
θ_1'	0.9982	0.9982	0.9739
θ_2'	0.9991	0.9991	0.9777
$V(\bar{y})$	163356.4086	163356.4086	27.1254
$V(\bar{x})$	6884.4455	6857.8555	2894.3089
$V(m)$	101518.7738	101518.7738	26.1307
$Cov(\bar{y}, m)$	90236.2939	90236.2939	21.0918
$Cov(\bar{y}, \bar{x})$	15061.4011	14905.0488	182.7425
ρ	0.4491	0.4453	0.6522

Table 4.1: Parameters and constants computed from the 3 populations

Parameters	For sample size $n = 5$		
	Popln-1	**Popln-2**	**Popln-3**
N	34	34	20
n	5	5	5
N_{C_n}	278256	278256	15504
$\bar{Y}$	856.4118	856.4118	41.5
$\bar{M}$	736.9811	736.9811	40.0552
M	767.5	767.5	40.5
$\bar{X}$	208.8824	199.4412	441.95
M_d	150	142.5	407.5
R	4.0999	4.2941	0.0939
R'	1.1158	1.1158	1.0247
θ_1'	0.9981	0.9981	0.9736
θ_2'	0.9990	0.9990	0.9775
$V(\bar{y})$	91690.3713	91690.3713	14.3605
$V(\bar{x})$	3864.1726	3849.248	1532.2812
$V(m)$	59396.2836	59396.2836	10.8348
$Cov(\bar{y}, m)$	48074.9542	48074.9542	9.0665
$Cov(\bar{y}, \bar{x})$	8453.8187	8366.0597	96.7461
ρ	0.4491	0.4453	0.6522

Table: 4.2. Parameters and constants computed from the 3 populations

Estimators		For sample size $n = 3$			For sample size $n = 5$		
		Popln-1	Popln-2	Popln-3	Popln-1	Popln-2	Popln-3
Existing	$\hat{\bar{Y}}_R$	63.0241	72.9186	0.2015	35.3748	40.9285	0.1067
Proposed	$\hat{\bar{Y}}_{SP\,1}$	51.7334	51.7334	0.3842	57.4970	57.4970	0.4856
	$\hat{\bar{Y}}_{SP\,2}$	51.9128	51.9128	0.3882	57.6265	57.6265	0.4886

Table: 4.3. Bias of the existing and proposed estimators

Estimators		For sample size $n = 3$			For sample size $n = 5$		
		Popln-1	Popln-2	Popln-3	Popln-1	Popln-2	Popln-3
Existing	$\hat{\bar{Y}}_r$	163356.4086	163356.4086	27.1254	91690.3713	91690.3713	14.3605
	$\hat{\bar{Y}}_R$	155577.8155	161802.8878	18.3261	87324.3215	90818.3961	9.7020
	$\hat{\bar{Y}}_{lr}$	130408.9222	130964.1249	15.5872	73197.2660	73508.8959	8.2520
Proposed	$\hat{\bar{Y}}_{SP\,1}$	88286.3450	88285.8793	11.0521	58283.5957	58283.2270	7.0552
	$\hat{\bar{Y}}_{SP\,2}$	88331.1066	88331.3470	11.0915	58319.0161	58319.2062	7.0690

Table: 4.4. Variance / Mean squared error of the existing and proposed estimators

The percentage relative efficiencies of the proposed estimators with respect to the existing estimators are also obtained and are given in the following tables:

Estimators	For sample size $n = 3$			For sample size $n = 5$		
	Popln-1	Popln-2	Popln-3	Popln-1	Popln-2	Popln-3
$\hat{\bar{Y}}_r$	185.03	185.03	245.43	157.32	157.32	203.54
$\hat{\bar{Y}}_R$	176.22	183.27	165.82	149.83	155.82	137.52
$\hat{\bar{Y}}_{lr}$	147.71	148.34	141.03	125.59	126.12	116.96

Table: 4.5. Percentage Relative Efficiency of Proposed Estimators

Estimators	For sample size $n = 3$			For sample size $n = 5$		
	Popln-1	Popln-2	Popln-3	Popln-1	Popln-2	Popln-3
$\hat{\bar{Y}}_r$	184.94	184.94	244.56	157.22	157.22	203.15
$\hat{\bar{Y}}_R$	176.13	183.18	165.23	149.74	155.73	137.25
$\hat{\bar{Y}}_{lr}$	147.64	148.26	140.53	125.51	126.05	116.74

Table: 4.6. Percentage Relative Efficiency of Proposed Estimators

From the Tables 4.5 and 4.6, it is observed that, the percentage relative efficiencies of the proposed estimators with respect to existing estimators are in general ranging from 116.74 to 245.43. Particularly, the PRE is ranging from 157.22 to 245.43

for comparing with SRSWOR sample mean; ranging from 137.25 to 183.27 for comparing with ratio estimator; and ranging from 116.74 to 148.34 for comparing with linear regression estimator. This shows that the proposed estimators perform better than the existing SRSWOR sample mean, ratio and linear regression estimator for all the three populations considered here. Further it is observed from the numerical comparison that the following inequalities are hold:

$$\mathrm{MSE}\left(\hat{\bar{Y}}_{\mathrm{SPj}}\right) \leq \mathrm{V}\left(\hat{\bar{Y}}_{\mathrm{lr}}\right) \leq \mathrm{MSE}\left(\hat{\bar{Y}}_{\mathrm{R}}\right) \leq \mathrm{V}\left(\hat{\bar{Y}}_{\mathrm{r}}\right)$$

5. Conclusion

This paper deals with two new median based modified ratio estimators using the linear combinations of population median and population mean of the auxiliary variable. The conditions are derived for which the proposed estimators are more efficient than the existing estimators. Further it is shown that the percentage relative efficiencies of the proposed estimators with respect to existing estimators are ranging from 116.74 to 245.43 for certain natural populations available in the literature. It is usually believed that the linear regression estimator is the best linear unbiased estimator or the optimum estimator for estimating the population mean whenever there exist an auxiliary variable, which is positively correlated with that of the study variable. However it is shown that the proposed median based modified ratio estimators are outperformed not only the ratio estimator but also the linear regression estimator. Hence the proposed median based modified ratio estimators are recommended for the practical problems.

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Appendix A

The derivation of the bias and the mean squared error of $\widehat{\bar{Y}}_{SP\,1}$ are given below:

Consider $\widehat{\bar{Y}}_{SP\,1} = \bar{y}\left[\frac{M_d\ M+\bar{X}}{M_d\ m+\bar{X}}\right]$ (A1)

Let $e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ and $e_1 = \frac{m-M}{M}$

$\Rightarrow E(e_0) = 0;\ E(e_1) = \frac{\text{Bias}\,(m)}{M}$ (A2)

$\Rightarrow E(e_0^2) = \frac{V(\bar{y})}{\bar{Y}^2}; E(e_1^2) = \frac{V(m)}{M^2}\ ; E(e_0 e_1) = \frac{Cov\,(\bar{y},m)}{\bar{Y}M}$ (A3)

The estimator $\widehat{\bar{Y}}_{SP\,1}$ can be written in terms of e_0 and e_1 as

$$\widehat{\bar{Y}}_{SP\,1} = \bar{Y}(1+e_0)\left(\frac{M_d\ M+\bar{X}}{M_d\ M(1+e_1)+\bar{X}}\right)$$

$$\Rightarrow \widehat{\bar{Y}}_{SP\,1} = \bar{Y}(1+e_0)\left[\frac{1}{1+\left(\frac{M_d\ M}{M_d\ M+\bar{X}}\right)e_1}\right]$$

$$\Rightarrow \hat{\bar{Y}}_{SP1} = \bar{Y}(1+e_0)\left(\frac{1}{1+\theta_1' e_1}\right) \text{; where } \theta_1' = \frac{M_d\, M}{M_d\, M + \bar{\bar{X}}}$$

$$\Rightarrow \hat{\bar{Y}}_{SP1} = \bar{Y}(1+e_0)(1+\theta_1' e_1)^{-1}$$

Neglecting the terms of higher order, we have

$$\hat{\bar{Y}}_{SP1} = \bar{Y}(1+e_0)(1-\theta_1' e_1 + \theta_1'^{\,2} e_1^{\,2})$$

$$\Rightarrow \hat{\bar{Y}}_{SP1} - \bar{Y} = \bar{Y}e_0 - \bar{Y}\theta_1' e_1 - \bar{Y}\theta_1' e_0 e_1 + \bar{Y}\theta_1'^{\,2} e_1^{\,2} \qquad \text{(A4)}$$

Taking expectations on both sides of (A4) we have,

$$E(\hat{\bar{Y}}_{SP1} - \bar{Y}) = \bar{Y}E(e_0) - \bar{Y}\theta_1' E(e_1) - \bar{Y}\theta_1' E(e_0 e_1) + \bar{Y}\theta_1'^{\,2} E(e_1^{\,2})$$

$$\Rightarrow E\left(\hat{\bar{Y}}_{SP1} - \bar{Y}\right) = \bar{Y}\left\{\theta_1'^{\,2} E(e_1^{\,2}) - \theta_1' E(e_0 e_1) - \theta_1' E(e_1)\right\} \quad \text{from (A2) and (A3)}$$

$$\Rightarrow \text{Bias}(\hat{\bar{Y}}_{SP1}) = \bar{Y}\left\{\theta_1'^{\,2} C_{mm}' - \theta_1' C_{ym}' - \theta_1' \frac{\text{Bias (m)}}{M}\right\} \qquad \text{(A5)}$$

The mean squared error of $\hat{\bar{Y}}_{SP1}$ is obtained as given below:

$$MSE\left(\hat{\bar{Y}}_{SP1}\right) = E(\hat{\bar{Y}}_{SP1} - \bar{Y})^2 = E(\bar{Y}e_0 - \bar{Y}\theta_1' e_1)^2$$

$$\Rightarrow MSE\left(\hat{\bar{Y}}_{SP1}\right) = \bar{Y}^2\left\{E(e_0^2) + \theta_1'^{\,2} E(e_1^{\,2}) - 2\theta_1' E(e_0 e_1)\right\}$$

$$\Rightarrow MSE\left(\hat{\bar{Y}}_{SP1}\right) = V(\bar{y}) + \frac{\bar{Y}^2}{M^2}\theta_1'^{\,2}\, V(m) - 2\frac{\bar{Y}}{M}\theta_1'\, Cov(\bar{y}, m)$$

$$\Rightarrow MSE\left(\hat{\bar{Y}}_{SP1}\right) = V(\bar{y}) + R'^2 \theta_1'^{\,2}\, V(m) - 2R'\theta_1'\, Cov(\bar{y}, m); \; R' = \frac{\bar{Y}}{M} \qquad \text{(A6)}$$

The bias and mean squared error of $\hat{\bar{Y}}_{SP2}$ can be obtained in a similar manner.

Appendix-B

The conditions for which the proposed estimators perform better than the existing estimators are derived here and are given below.

Comparison with that of SRSWOR sample mean

Consider $MSE(\hat{\bar{Y}}_{SPj}) \leq V(\hat{\bar{Y}}_r)$

$$\Rightarrow V(\bar{y}) + R'^2 \theta_j'^{\,2}\, V(m) - 2R'\theta_j'\, Cov(\bar{y}, m) \leq V(\bar{y})$$

$$\Rightarrow R'^2 \theta_j'^{\,2}\, V(m) - 2R'\theta_j'\, Cov(\bar{y}, m) \leq 0$$

$$\Rightarrow R'^2 \theta_j'^{\,2}\, V(m) \leq 2R'\theta_j'\, Cov(\bar{y}, m)$$

$$\Rightarrow Cov(\bar{y}, m) \geq \frac{R'\theta_j'\, V(m)}{2}$$

$$\Rightarrow Cov(\bar{y}, m) \geq \frac{\bar{Y}M\theta_j'\, C_{mm}'}{2}$$

$$\Rightarrow 2C_{ym}' \geq \theta_j'\, C_{mm}' \;; j = 1, 2$$

Comparison with that of Ratio Estimator

Consider $MSE(\hat{\bar{Y}}_{SPj}) \leq MSE(\hat{\bar{Y}}_R)$

$$\Rightarrow V(\bar{y}) + R'^2 \theta_j'^{\,2}\, V(m) - 2R'\theta_j'\, Cov(\bar{y}, m) \leq V(\bar{y}) + R^2 V(\bar{x}) - 2R Cov(\bar{y}, \bar{x})$$

$$\Rightarrow R'^2 \theta_j'^{\,2}\, V(m) - 2R'\theta_j'\, Cov(\bar{y}, m) \leq R^2 V(\bar{x}) - 2R Cov(\bar{y}, \bar{x})$$

$$\Rightarrow R'^2 \theta_j'^{\,2}\, V(m) - R^2 V(\bar{x}) \leq 2R'\theta_j'\, Cov(\bar{y}, m) - 2R Cov(\bar{y}, \bar{x})$$

$$\Rightarrow \frac{\overline{Y}^2}{M^2}\theta_j^{'2}\, V(m) - \frac{\overline{Y}^2}{\overline{X}^2} V(\bar{x}) \le 2\frac{\overline{Y}}{M}\theta_j^{'}\, Cov(\bar{y}, m) - 2\frac{\overline{Y}}{\overline{X}} Cov(\bar{y}, \bar{x})$$

$$\Rightarrow \theta_j^{'2}\, \frac{V(m)}{M^2} - \frac{V(\bar{x})}{\overline{X}^2} \le 2\left\{\theta_j^{'}\, \frac{Cov(\bar{y}, m)}{\overline{Y}M} - \frac{Cov(\bar{y}, \bar{x})}{\overline{Y}\overline{X}}\right\}$$

$$\Rightarrow \theta_j^{'2}\, C_{mm}^{'} - C_{xx}^{'} \le 2\left\{\theta_j^{'}\, C_{ym}^{'} - C_{yx}^{'}\right\}; j = 1, 2$$

Comparison with that of Linear Regression Estimator

Consider $MSE(\hat{\hat{Y}}_{SPj}) \le V(\hat{\hat{Y}}_{lr})$

$$\Rightarrow V(\bar{y}) + R^{'2}\theta_j^{'2}\, V(m) - 2R^{'}\theta_j^{'}\, Cov(\bar{y}, m) \le V(\bar{y})(1 - \rho^2)$$

$$\Rightarrow R^{'2}\theta_j^{'2}\, V(m) - 2R^{'}\theta_j^{'}\, Cov(\bar{y}, m) \le -V(\bar{y})\left(\frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x}) * V(\bar{y})}\right)$$

$$\Rightarrow 2R^{'}\theta_j^{'}\, Cov(\bar{y}, m) - R^{'2}\theta_j^{'2}\, V(m) \ge \frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x})}$$

$$\Rightarrow 2\frac{\overline{Y}}{M}\theta_j^{'}\, Cov(\bar{y}, m) - \frac{\overline{Y}^2}{M^2}\theta_j^{'2}\, V(m) \ge \frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x})}$$

$$\Rightarrow 2\overline{Y}^2\theta_j^{'}\, C_{ym}^{'} - \overline{Y}^2\theta_j^{'2}\, C_{mm}^{'} \ge \frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x})}$$

$$\Rightarrow 2\theta_j^{'}\, C_{ym}^{'} - \theta_j^{'2}\, C_{mm}^{'} \ge \frac{[C_{yx}^{'}]^2}{C_{xx}^{'}}\ ; j = 1, 2$$