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DISCRIMINATION BETWEEN LINEAR FAILURE RATE DISTRIBUTION AND RAYLIEGH DISTRIBUTION

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Abstract

The purpose of the study is to suggest test statistics for discriminating between linear failure rate distribution (LFRD) and Rayleigh distribution while considering these distributions as null and alternative populations respectively. To test statistics based on moments of order statistics and population quantiles are proposed and their percentiles are evaluated. The performance of the test procedures are also compared through computed power functions.

Kew Words: Linear Failure Rate Distribution, Rayleigh Distribution, Order Statistics, Quantiles, Power Function.

1. Introduction

In reliability studies series systems are one of many popular system configurations. If a series system has two components having independently distributed life time random variables with failure rate functions $h_1(x)$ and $h_2(x)$then it is well known that the reliability of the series system is

$$R(x) = exp\left[-\int_0^x \{h_1(t) + h_2(t)\}dt\right] \quad (1)$$

The corresponding cumulative distribution function, failure density function and failure rate function are respectively given by

$$F(x) = 1 - exp\left[-\int_0^x \{h_1(t) + h_2(t)\}dt\right] \quad (2)$$

$$f(x) = \frac{d}{dx}F(x), \quad (3)$$

$$h(x) = \frac{f(x)}{R(x)}. \quad (4)$$

Taking $h_1(x), h_2(x)$, as the failure rates of the well known exponential and Rayleigh distributions in (1) we get the most commonly used Linear Failure Rate Distribution (LFRD). More specifically, if $h_1(x) = a$and $h_2(x) = bx$where (a>0,b>0) we get, the failure density function, cumulative distribution function, hazard or failure rate function of LFRD as:

$$f(x) = (a + bx)e^{-\left(ax+\frac{bx^2}{2}\right)};\ x > 0, a > 0, b > 0, \quad (5)$$

$$F(x) = 1 - e^{-\left(ax+\frac{bx^2}{2}\right)}; x > 0, a > 0, b > 0, \quad (6)$$

$$h(x) = a + bx. \quad (7)$$

This distribution has non-zero density at the origin, so that it may be of important use in connection with those types of responses which take place even before observation begins. Listings of similar response time densities are given in Barlow and Proschan (1965). In that sense *h(x)* is also called the conditional mortality rate if response time is survival time. In the context of competing risks LFRD is the distribution of the minimum of two independent random variables of which one follows exponential and the other follows Rayleigh distribution.

Bain (1974) seems to be one of the earliest works that has touched upon LFRD as a model useful for analysis in life testing.

Some basic features of LFRD are as follows:

Mean:

$$\mu = \sqrt{\frac{2\pi}{b}} e^{a^2/2b}\left(1 - \emptyset(a/\sqrt{b})\right), \quad (8)$$

where $\emptyset(.)$ denotes the cumulative distribution function of a standard normal variate.

Variance:

$$\sigma^2 = \frac{2}{b}(1 - a\mu) - \mu^2, \quad (9)$$

Mode:

$$M = \left(\sqrt{\frac{1}{b}} - \frac{a}{b}\right) I(a^2 < b), \quad (10)$$

where I(.) denotes indicator function.

100 p^{th} Percentile:

$$F^{-1}(p) = \sqrt{\left(\frac{a}{b}\right)^2 - \frac{2\log(1-p)}{b}} - \frac{a}{b}, \quad (11)$$

and hence median is

$$M_d = \sqrt{\left(\frac{a}{b}\right)^2 - \frac{2\log(0.5)}{b}} - \frac{a}{b}, \quad (12)$$

In biological sciences this is called 50% survival time denoted by t_{50}.

Recurrence relation for raw moments is

$$\mu_k^1 = \frac{a}{k+1}\mu'_{k+1} + \frac{b}{k+2}\mu'_{k+2}; k = 0,1,2 .... \quad (13)$$

The second, third and fourth raw moments are

$$\mu'_2 = \frac{2}{b}(1 - a\mu), \quad (14)$$

$$\mu'_3 = \frac{3}{b}\left(\mu - \frac{a}{b}(1 - a\mu)\right) \quad (15)$$

$$\mu'_4 = \frac{8}{b^2} + \frac{4a^2}{b^3} - \mu\left(\frac{12a}{b^2} + \frac{4a^3}{b^3}\right), \quad (16)$$

where μ is the mean of the distribution given by (8).

It can be seen from (10) that LFRD has a non-zero mode only if its parameters '*a*' and '*b*' satisfy the relation $a^2 < b$ with *a*> 0, *b*> 0.

The graphs of density function of LFRD for various combinations of the parameters '*a*', '*b*' are shown in the following figures.

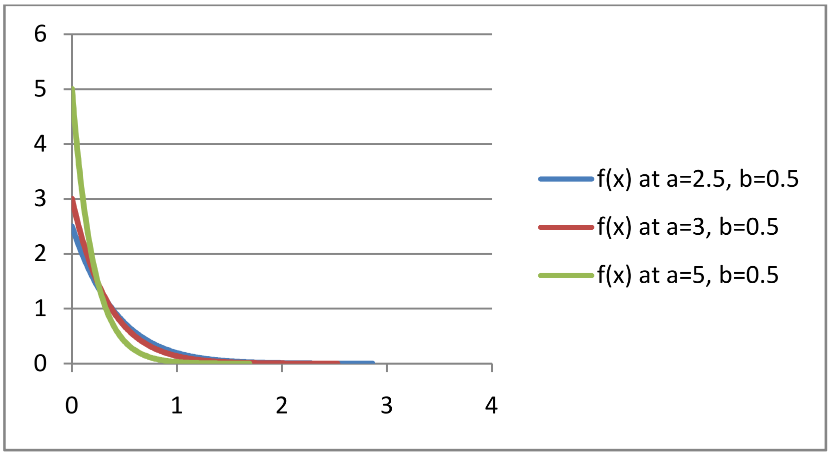

Fig. 1: Density function of LFRD for (a=2.5, b=0.5), (a=3, b=0.5), (a=5, b=0.5)

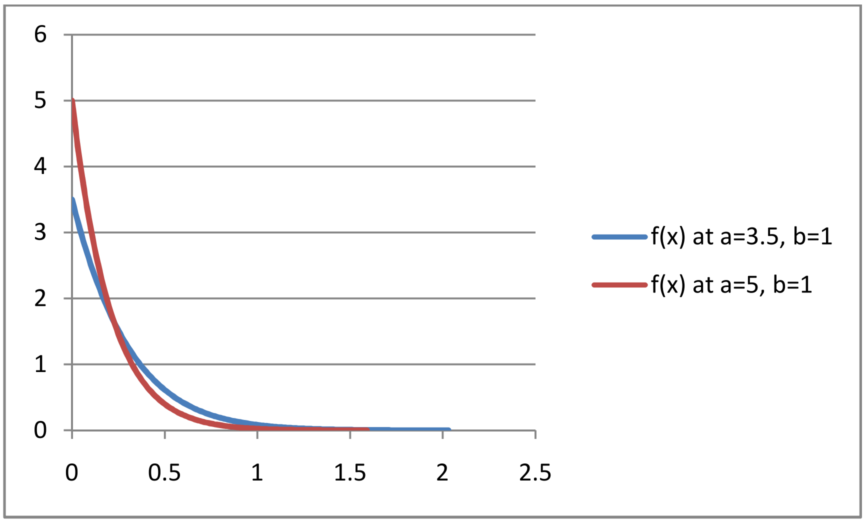

Fig. 2: Density function of LFRD for (a=3.5, b=1), (a=5, b=1)

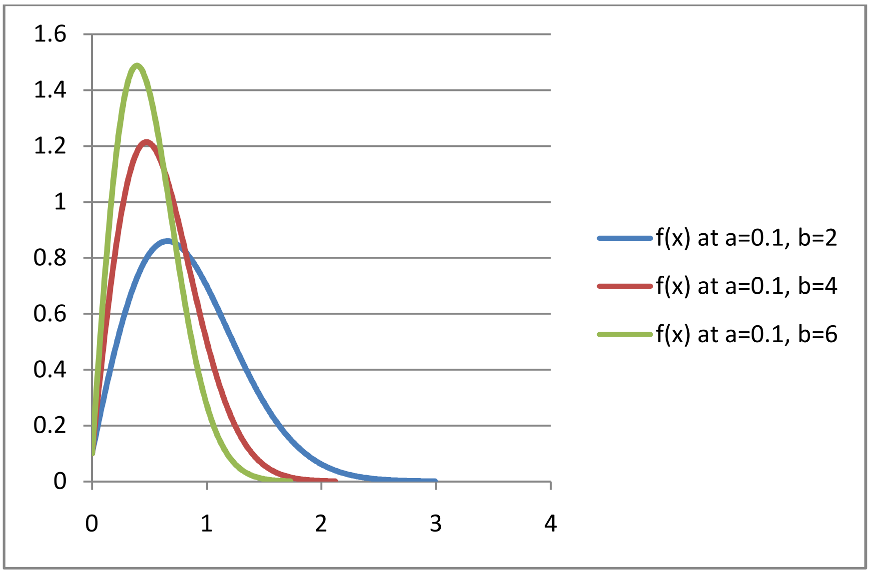

Fig. 3: Density function of LFRD for (a=0.1, b=2), (a=0.1, b=4), (a=0.1, b=6)

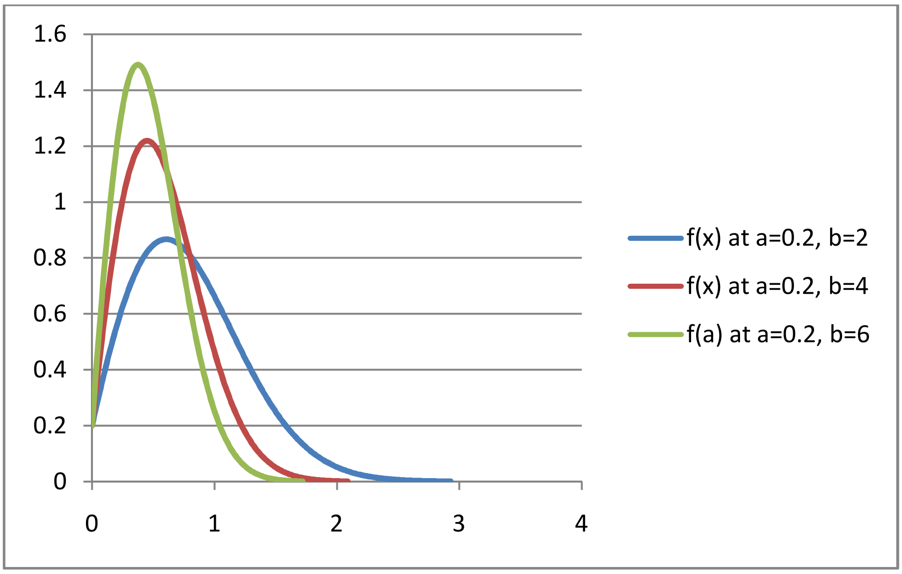

Fig. 4: Density function of LFRD for (a=0.2, b=2), (a=0.2, b=4), (a=0.2, b=6)

In Figure 1 and 2,the combinations of 'a' and 'b' are bound by$a^2 > b$, accordingly the mode is zero and the graphs are similar to that of exponential distribution. On the other hand for figure 3,4 are the parameters satisfy $a^2 < b$resulting in the respective non-zero modes. These chief characteristics of LFRD and its component distributions - exponential and Rayleigh motivated us to study the discriminatory aspect between LFRD and exponential/ Rayleigh through statistical test procedures. Such studies of discriminatory problems between probability models are made by Gupta *et al.* (2002), Gupta and Kundu (2003a), Gupta and Kundu (2003b), Kundu and Gupta (2004),Kundu and Raqab (2007), Arabin and Kundu (2009), Arabin and Kundu (2012) and the references therein. Recently Sultan (2007) developed a test criterion to distinguish generalized exponential distribution from Weibull, Normal distributions using moments of order statistics in samples drawn from generalized exponential distribution. In this paper we adopt the criterion suggested by Sultan (2007) and propose another test procedure based on quantiles of LFRD. LFRD is considered as Null population (P_0), Rayleigh is considered as an alternative population (P_1). The rest of the paper is organised as follows. A brief description of the procedure developed by Sultan (2007) and its application to our models is presented in Section 2. Our proposed test based on population quantile is given in Section 3. The powers of the test procedures and their comparison are given in Section 4.

2. LFRD Vs Rayleigh Distribution using Moments of Order Statistics

Let $x_1, x_2, .. x_n$ be a random sample of size n. Here we test the Null hypothesis
H_0: The sample has come from LFRD against the alternative hypothesis
H_1: The sample has come from Rayleigh distribution
Sultan (2007) suggested a test statistic given by the formula

$$T = \frac{\sum_{i=1}^{n} x_{(i)}\alpha_i}{\sqrt{\sum_{i=1}^{n} x_{(i)}^2 \sum_{i=1}^{n} \alpha_i^2}}, \tag{17}$$

where $x_{(i)}$ - i^{th} ordered observation in the sample.
α_i - the expected value of i^{th} standard order statistic in a sample of size n from the null population.

If $x_1, x_2, .. x_n$ is truly a sample from the null population the formula for '*T*' would serve as a test statistic to discriminate a null population and the corresponding alternative population with the help of its critical values.

Hence, the sampling distribution of '*T*' and its percentiles therefrom are essential to make use of the test statistic '*T*'. In the present situation, because the null population is LFRD we need α_i from samples of LFRD which are available in Bakheet (2006). We borrow these α_i to develop the percentiles of '*T*'. However, '*T*' is an expression of non-linear terms of order statistics. Hence it is not feasible to work out for its sampling distribution analytically. We therefore, evaluated the empirical percentiles of '*T*' for LFRD as follows:

In Bakheet (2006), α_i are available for *n*=5, (*a*=0.1, *b*=2), (*a*=0.1, *b*=4), (*a*=0.1, *b*=6), (*a*=0.2, *b*=2), (*a*=0.2, *b*=4) and (*a*=0.2, *b*=6) in numerically evaluated form. We have generated 10,000 random samples of size 5 from LFRD with parametric combinations (*a*=0.1, *b*=2), (*a*=0.1, *b*=4), (*a*=0.1, *b*=6), (*a*=0.2, *b*=2), (*a*=0.2, *b*=4), (*a*=0.2, *b*=6) and evaluated '*T*' for each ordered sample. Selected cut off points of '*T*' are presented in the following Table -1 for *n*=5.

a	b	p \ n	0.001	0.00135	0.0027	0.005	0.01	0.025	0.05	0.1
0.1	2	5	0.76334	0.77082	0.78667	0.79709	0.81672	0.8442	0.8677	0.89355
0.1	4	5	0.76925	0.77604	0.7921	0.80226	0.82113	0.84838	0.8718	0.89665
0.1	6	5	0.77129	0.77862	0.79416	0.80426	0.82318	0.8504	0.87382	0.89831
0.2	2	5	0.74395	0.75376	0.76611	0.78124	0.80155	0.83012	0.85494	0.88321
0.2	4	5	0.75505	0.76421	0.77663	0.79013	0.81024	0.83813	0.86228	0.88917
0.2	6	5	0.76334	0.77082	0.78667	0.79709	0.81672	0.8442	0.8677	0.89355
		P \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
0.1	2	5	0.99285	0.99611	0.99767	0.99885	0.99919	0.99944	0.99961	0.99965
0.1	4	5	0.99319	0.99632	0.99774	0.99889	0.99921	0.99946	0.99959	0.99963
0.1	6	5	0.99338	0.9964	0.99777	0.99891	0.99922	0.99946	0.9996	0.99964
0.2	2	5	0.99147	0.99541	0.9974	0.9987	0.99912	0.99934	0.99954	0.99959
0.2	4	5	0.99228	0.99584	0.99757	0.9988	0.99916	0.9994	0.99956	0.99963
0.2	6	5	0.99285	0.99611	0.99767	0.99885	0.99919	0.99944	0.99961	0.99965

Table - 1: Percentiles of T Using Moments of Order Statistics

The percentiles of '*T*' would serve as critical values to test null hypothesis that a given sample of size n=5, comes from LFRD.

3. LFRD Vs Rayleigh Distribution using Population Quantiles

Expected values of the order statistics of LFRD are available in numerical form for *n*= 2 (1) 5 only. Hence, we propose a statistic similar to '*T*' based on population quantilesin place of moments of order statistics. Our proposed statistic is

$$T^* = \frac{\sum_{i=1}^{n} \delta_i x_{(i)}}{\sqrt{\sum_{i=1}^{n} x_{(i)}^2 \sum_{i=1}^{n} \delta_i^2}}, \quad (18)$$

where $\delta_i = F^{-1}(p) = \sqrt{\left(\frac{a}{b}\right)^2 - \frac{2\log(1-p_i)}{b}} - \frac{a}{b}$ with $p_i = \frac{i}{n+1}$.

We have tabulated the percentiles of empirical sampling distributions of 'T^*' for *n*=5 (5) 20, (*a*=0.1, *b*=2), (*a*=0.1, *b*=4), (*a*=0.1, *b*=6), (*a*=0.2, *b*=2), (*a*=0.2, *b*=4), (*a*=0.2, *b*=6) through 10,000 Monte-Carlo simulation runs and are given in Table 2.

a	b	P \ n	0.001	0.00135	0.0027	0.005	0.01	0.025	0.05	0.1
0.1	2	5	0.40644	0.40929	0.4343	0.44706	0.46998	0.50303	0.53483	0.57441
		10	0.58425	0.59901	0.62019	0.64228	0.66244	0.69022	0.717	0.74884
		15	0.62114	0.62911	0.653	0.67318	0.69645	0.7253	0.74993	0.77794
		20	0.6271	0.63808	0.65902	0.68001	0.7007	0.7313	0.76267	0.79112
		P \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
		5	0.81026	0.83355	0.85373	0.87167	0.8814	0.89043	0.89498	0.90006
		10	0.92571	0.93858	0.94779	0.95778	0.96159	0.96404	0.96687	0.96773
		15	0.94532	0.95649	0.96419	0.97107	0.97378	0.97629	0.97769	0.97855
		20	0.95212	0.96274	0.96962	0.97521	0.97834	0.98013	0.98178	0.98254

a	b	p \ n	0.001	0.00135	0.0027	0.005	0.01	0.025	0.05	0.1
0.1	4	5	0.41737	0.42075	0.44602	0.45846	0.48041	0.5137	0.54532	0.58452
		10	0.59449	0.61282	0.63325	0.65312	0.67216	0.70079	0.72566	0.75672
		15	0.63176	0.64085	0.66508	0.68576	0.7071	0.73509	0.75931	0.78662
		20	0.63767	0.65399	0.67392	0.69226	0.71402	0.74241	0.77259	0.80039
		p \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
		5	0.81552	0.83825	0.85787	0.87533	0.88473	0.89359	0.89789	0.90289
		10	0.92821	0.94047	0.94931	0.95888	0.96266	0.96505	0.96773	0.96853
		15	0.94729	0.9579	0.96541	0.97191	0.97447	0.97692	0.97827	0.97923
		20	0.95416	0.96424	0.97073	0.97612	0.9791	0.98074	0.98233	0.98308

a	b	P \ n	0.001	0.00135	0.0027	0.005	0.01	0.025	0.05	0.1
0.1	6	5	0.42239	0.42654	0.45119	0.46405	0.48561	0.51911	0.54986	0.58905
		10	0.59962	0.61779	0.63951	0.65663	0.67634	0.70497	0.73007	0.76076
		15	0.63592	0.64642	0.67114	0.69185	0.71233	0.73965	0.76368	0.79056
		20	0.64525	0.6622	0.68125	0.6971	0.71935	0.74776	0.77694	0.80436
		P \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
		5	0.81778	0.84041	0.85956	0.87699	0.88631	0.89494	0.89917	0.90414
		10	0.92935	0.94138	0.94996	0.95939	0.96312	0.9655	0.96814	0.96888
		15	0.94819	0.95856	0.96596	0.97233	0.97477	0.9772	0.97853	0.97953
		20	0.95508	0.96492	0.97125	0.97647	0.97942	0.98104	0.98258	0.98332

a	b	P \ n	0.001	0.00135	0.0027	0.005	0.01	0.025	0.05	0.1
0.2	2	5	0.37181	0.37643	0.39863	0.41548	0.43547	0.46879	0.50205	0.54135
		10	0.55412	0.56122	0.5889	0.61367	0.63268	0.6604	0.69052	0.72336
		15	0.5895	0.59214	0.62338	0.64345	0.66783	0.69638	0.72385	0.75401
		20	0.58513	0.59971	0.62045	0.64634	0.67036	0.70322	0.73389	0.76658
		P \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
		5	0.79267	0.81735	0.83932	0.85984	0.87022	0.87965	0.88509	0.89041
		10	0.91697	0.93196	0.94239	0.95366	0.95786	0.96075	0.96377	0.96495
		15	0.93823	0.95131	0.9603	0.96795	0.97136	0.9737	0.97566	0.97655
		20	0.9452	0.95745	0.96571	0.97232	0.97584	0.97772	0.98001	0.9808

a	b	P \ n	0.001	0.00135	0.0027	0.005	0.01	0.025	0.05	0.1
0.2	4	5	0.39057	0.3952	0.41919	0.43444	0.45542	0.48807	0.52072	0.56025
		10	0.57138	0.58038	0.60389	0.62734	0.64901	0.67716	0.70561	0.7378
		15	0.60504	0.61287	0.63967	0.65948	0.68369	0.71203	0.73793	0.76738
		20	0.60811	0.62316	0.64184	0.66559	0.68694	0.71777	0.74979	0.77977
		p \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
		5	0.80299	0.8268	0.84772	0.86662	0.87686	0.88595	0.89088	0.89605
		10	0.92187	0.9358	0.94564	0.95606	0.96004	0.96266	0.96557	0.96659
		15	0.94237	0.95444	0.96261	0.96967	0.9728	0.97532	0.97686	0.9777
		20	0.94918	0.96058	0.96804	0.97405	0.97734	0.97909	0.98105	0.9818

a	b	P \ n	**0.001**	**0.00135**	**0.0027**	**0.005**	**0.01**	**0.025**	**0.05**	**0.1**
0.2	6	5	0.39929	0.40404	0.42843	0.44174	0.46458	0.49695	0.52947	0.56893
		10	0.57902	0.59229	0.61383	0.63683	0.65721	0.68524	0.7129	0.74456
		15	0.61249	0.62247	0.6479	0.66754	0.69109	0.72027	0.74545	0.77384
		20	0.6194	0.63116	0.65285	0.67624	0.69592	0.72579	0.7576	0.78695
		P \ n	**0.9**	**0.95**	**0.975**	**0.99**	**0.995**	**0.9973**	**0.99865**	**0.99**
		5	0.80755	0.831	0.85148	0.86974	0.87971	0.88876	0.89345	0.89856
		10	0.92423	0.93752	0.94699	0.95715	0.96101	0.9635	0.96638	0.9673
		15	0.94427	0.95563	0.96359	0.9705	0.97342	0.97595	0.97738	0.97819
		20	0.95103	0.96196	0.96903	0.97473	0.97806	0.97971	0.98151	0.98226

Table -2: Percentiles of T* Using Quantiles

The percentiles of 'T' or 'T^*' would serve as critical values to test null hypothesis that a given sample comes from LFRD.

4. Comparative Study

The power of these test statistics with Rayleigh as alternative is evaluated as follows:

10,000 random samples of size n= 5 (5) 20 from Rayleigh (a=1, b=0) are generated and the values of 'T', 'T^*'are calculated at each sample retaining α_i of 'T' or $\delta_i$of 'T^*' as those of the corresponding LFRD (which is the null population here) only. The proportions of values of 'T', 'T^*' that fall above a specified percentile, say 95th out of 10,000 are noted down which represent the power of the statistics 'T' or 'T^*'. These are given in Table - 3 for n=5 and in Table - 4 for other values of n=10,15,20.

Parameter Combinations	***n*=5**
a=0.1, b=2	0.0604
a=0.1, b=4	0.0564
a=0.1, b=6	0.0549
a=0.2, b=2	0.0732
a=0.2, b=4	0.0664
a=0.2, b=6	0.0618

Table -3: Powers of *T* Statistic at α = 0.95 based on Moments of Order Statistics (Rayleigh Alternative)

Parameter Combinations	***n*=5**	***n*=10**	***n*=15**	***n*=20**
a=0.1, b=2	0.0873	0.0778	0.0756	0.0773
a=0.1, b=4	0.0744	0.0701	0.068	0.0676
a=0.1, b=6	0.0683	0.0669	0.0651	0.0634
a=0.2, b=2	0.1348	0.1107	0.1057	0.1096
a=0.2, b=4	0.1048	0.0922	0.0868	0.0903
a=0.2, b=6	0.0939	0.0835	0.0804	0.0823

Table - 4: Power of T^* Statistic at α = 0.95 based on Quantiles

Table 4 reveals that at n=5 the power of 'T^*' is more than that of 'T' at 5% level of significance. Had the moments of order statistics been available the same trend of 'T^*' over 'T' might have been noticed. However, adopting the same procedure of rejection we found that the construction of 'T^*' would be useful to test the other hypotheses also. Accordingly Tables - 3 and Table - 4 would be the indicator of performance of the test formula. This would be useful to develop test statistics for the other combinations of n. Because of lack of availability of moments of order statisticsevaluated in published form, one can depend on quantiles and such dependencemay give more accurate results. We therefore suggest that 'T^*' can be used as an appropriate test statistic.

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