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BAYESIAN ESTIMATION IN PARETO TYPE-I MODEL

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Abstract

Bayes estimators of the shape parameters of a Pareto type-I model are obtained for different priors using Square Error and Asymmetric Precautionary Error Loss Functions through direct method and Lindley's approach. Bayes estimators of reliability and hazard rate functions have also been discussed. The calculations have been illustrated with the help of numerical example. Comparison between Square Error and Asymmetric Precautionary Error Loss Functions have also been shown with the help of a numerical example.

Key words: Bayes estimator, Maximum Likelihood Estimator, Prior, Reliability Function, Hazard Rate Function, Square Error Loss Function and Asymmetric Precautionary Error Loss Function.

1. Introduction

The Pareto distribution, named after the Italian Economist Vilfredo Pareto, is a power law probability distribution that coincides with social, scientific, geophysical, actuarial, and many other types of observable phenomena. Outside the field of economics, it is sometimes referred to as the Bradford distribution. The Pareto distribution was originally used to describe the allocation of wealth among individuals since it seemed to show rather well the way that a larger portion of the wealth of any society is owned by a smaller percentage of the people in that society. The probability density function of Pareto type – I distribution is given by

$$f(x,\theta,p)=\frac{p\theta^{p}}{x^{p+1}} \qquad p,\theta>0\ , \quad \theta<x<\infty$$
$$= 0 \qquad otherwise \tag{1}$$

The first parameter marks a lower bound on the possible values that a Pareto distributed random variable can take on.

Bayesian statistics provide a conceptually simple process for updating uncertainty in the light of evidence. Initial beliefs about some unknown quantity are represented by a prior distribution. Information in the data is expressed by the likelihood function. The prior distribution and the likelihood function are then combined to obtain the posterior distribution for the quantity of interest. The posterior distribution expresses our revised uncertainty in light of the data.

Ashour et al.(1994) used the quasi-likelihood function to derive Bayesian and non-Bayesian estimates for the unknown parameters of the Pareto distribution. Howlader and Hossain (2002) presented Bayesian estimation of the survival function of the Pareto distribution of the second kind using the methods of Lindley (1980) and Tierney and Kadane (1986). Bermudez and Turkman (2003) used several methods for estimating the parameters of the generalized Pareto distribution (GPD), namely maximum likelihood (ML), the method of moments (MOM) and the probability-weighted moments (PWM). Nigma and Hamdy (2007) supposed that the length of time in years for which a business operates until failure has a Pareto distribution. Pandey and Rao (2008) obtained Bayes estimators of the shape parameter of the generalized Pareto distribution by taking quasi, inverted gamma and uniform prior distributions using the linex, precautionary and entropy loss functions. These were compared with the corresponding Bayes estimators under the squared error loss function. Shukla and Kumar (2009) obtained Bayes estimators of the shape parameters of a generalized gamma type model for different priors using Lindley's approach.

In this paper, we have obtained Bayes estimators of the shape parameter p for fixed θ of a Pareto type-I model for different priors viz. uniform, Jeffrey's, exponential, Mukherjee-Islam's, Weibull, gamma etc. using Square Error and Asymmetric Precautionary Error Loss Functions through direct method and Lindley's (1980) approach. Bayes estimators of reliability and hazard rate functions have also been discussed.

2. Maximum likelihood estimator of p (for fixed θ)

The likelihood function is given by

$$l(p/\underset{\sim}{x}) = \frac{p^n}{\prod_{i=1}^{n} x_i} \exp[-p\sum_{i=1}^{n}\log(x_i/\theta) \tag{2}$$

Therefore, M.L.E. of p

$$\hat{p} = \frac{n}{\sum_{i=1}^{n}\log\frac{x_i}{\theta}} \tag{3}$$

3. Lindley's Approach

Bayes estimators are often obtained as the ratio of two integrals which cannot be solved by using asymptotic expansion and calculus of difference. Lindley (1980) developed an asymptotic approximation to the ratio

$$I = \frac{\int_p h(p)\,l(p/x)\,g(p)\,dp}{\int_p l(p/x)\,g(p)\,dp} \tag{4}$$

According to him

$$1 \approx h(p^*) + \frac{\sigma^2}{2}\left[h_2(p^*) + 2h_1(p^*)u_1(p^*)\right] + \frac{\sigma^4}{2}\left[L_3(p^*)h_1(p^*)\right] \tag{5}$$

where p^* is the MLE of p
Also,

$$L_k(p^*) = \frac{\partial^k}{\partial p^k} L(p)\Big|_{p=p^*} \tag{6}$$

$$h_k(p^*) = \frac{\partial^k}{\partial p^k} h(p)\Big|_{p=p^*} \tag{7}$$

$$\sigma^2 = -L_2^{-1}(p^*) \tag{8}$$

$$u(p^*) = \log g(p)\Big|$$

3. Bayes Estimators of p Given θ under Different Priors

If θ is fixed quantity, Bayes estimators of p may be obtained by using direct method and Lindley's approach as follows:
Here,

$$h(p) = p,\ h(p^*) = p^*,\ h_1(p^*) = 1,\ h_2(p^*) = 0$$

Bayes estimators (p_B) of p given θ for different priors using the abovesaid two methods are given in Table 1.

4. Bayes estimators of Reliability Function under Different Priors

Reliability is the probability of a device performing its purpose adequately for the period intended under the given operating conditions. If X is the lifetime of the unit, the reliability of the unit at time t is given by,

$$R(t) = P(X \geq t) = 1 - F(t)$$

Where F is the d.f. of the failure time X.

Reliability Function of the given model is given by

$$R(t) = P(X > t) = \int_t^{\infty} f(x)dx$$

$$= \int_t^{\infty} \frac{p\theta^p}{x^{p+1}} \qquad p, \theta > 0 \ and \ \ \theta < x < \infty$$

$$= \left(\frac{\theta}{t}\right)^p \tag{9}$$

Bayes estimators ($R(t)_p^B$) of reliability function given θ by using different priors are given in Table-2.

Prior	**Density**	p_B **Direct Method**	p_B **Lindley's Approach**
Uniform	$I_{(0,1)}(p)$	$\dfrac{n+1}{\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)}$	$\left[1+\frac{1}{n}\right]p^*$
Jeffrey's	$p^{-1}I_{(1,e)}(p)$	$\dfrac{n}{\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)}$	p^*
Exponential	$e^{-p}\,;\quad p>0$	$\dfrac{(n+1)}{\left(\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)+1\right)}$	$\left[1+\frac{1}{n}\right]p^*-\frac{p^{*2}}{n}$
Mukherjee-Islam	$\alpha\sigma^{-\alpha}p^{\alpha-1}I_{(0,\sigma)}(p)$; $\alpha,\sigma>0$	$\dfrac{(n+\alpha)}{\left(\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)\right)}$	$\left[1+\frac{\alpha}{n}\right]p^*$
Gamma	$\frac{1}{\sigma^{\alpha}\Gamma\alpha}p^{\alpha-1}e^{-p/\sigma}$; $\alpha,\sigma>0\,,p>0$	$\dfrac{(n+\alpha)}{\left(\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)+\frac{1}{\sigma}\right)}$	$\left[1+\frac{\alpha}{n}\right]p^*-\frac{p^{*2}}{n\sigma}$

Table 1 : The values of p^B under direct method and Lindley's Approach

Prior	$R(t)_p^{\,B}$
Uniform	$R^*(t)\left[1+\frac{p^{*2}}{2n}\left(\log_e\left(\frac{\theta}{t}\right)\right)^2+\frac{p^*}{n}\log_e\left(\frac{\theta}{t}\right)\right]$
Jeffrey's	$R^*(t)\left[1+\frac{1}{2}\left\{\left(\log_e\left(\frac{\theta}{t}\right)\right)^2-\frac{2}{p^*}\log_e\left(\frac{\theta}{t}\right)\right\}\frac{p^{*2}}{n}+\frac{p^*}{n}\log_e\left(\frac{\theta}{t}\right)\right]$
Exponential	$R^*(t)\left[1+\frac{1}{2}\left\{\left(\log_e\left(\frac{\theta}{t}\right)\right)^2-2\log_e\left(\frac{\theta}{t}\right)\right\}\frac{p^{*2}}{n}+\frac{p^*}{n}\log_e\left(\frac{\theta}{t}\right)\right]$
Mukherjee-Islam	$R^*(t)\left[1+\frac{1}{2}\left\{\left(\log_e\left(\frac{\theta}{t}\right)\right)^2+2\log_e\left(\frac{\theta}{t}\right)\frac{(\alpha-1)}{p}\right\}\frac{p^{*2}}{n}+\frac{p^*}{n}\log_e\left(\frac{\theta}{t}\right)\right]$

Prior	$R(t)_p^B$
Gamma	$R^*(t)\left[1+\frac{1}{2}\left\{\left(\log_e\left(\frac{\theta}{t}\right)\right)^2+2\log_e\left(\frac{\theta}{t}\right)\left(\frac{\alpha-1}{p^*}-\frac{1}{\sigma}\right)\right\}\frac{p^{*2}}{n}+\frac{p^*}{n}\log_e\left(\frac{\theta}{t}\right)\right]$

Table 2: Bayes estimator ($R(t)_p^B$) of Reliability Function given θ

5. Bayes estimators of Hazard Rate Function under Different Priors

The hazard function is a measure of the tendency to fail; the greater the value of the hazard function, the greater the probability of impending failure. Mathematically, the hazard function is defined as the ratio of the probability density function to the survival function , given by

$$H(x) = \frac{f(x)}{R(x)}$$

Hazard Rate Function H(t) of the given model is given by

$$H(t) = \frac{f(t)}{R(t)} = \frac{p}{t} \qquad (10)$$

Bayes estimators ($H(t)_p^B$) of Hazard rate function given θ by using different priors are given in Table-2.

Prior	$H(t)_p^B$
Uniform	$\frac{p^*}{t}\left(1+\frac{1}{n}\right)$
Jeffrey's	$\frac{p^*}{t}$
Exponential	$\frac{p^*}{t}\left(1+\frac{1}{n}\right)-\frac{p^{*2}}{nt}$
Mukharjee-Islam	$\frac{p^*}{t}\left(1+\frac{\alpha}{n}\right)$
Gamma	$\frac{p^*}{t}+\frac{p^{*2}}{nt}\left(\frac{\alpha-1}{p^*}-\frac{1}{\sigma}\right)+\frac{p^*}{nt}$

Table 3: Bayes estimator ($H(t)_p^B$) of Hazard Rate Function given θ

6. Loss Function

In the theory of point estimation, a loss function quantifies the losses associated to the errors committed while estimating a parameter. Often the expected value of the loss, called statistical risk, is used to compare two or more estimators: in such comparisons, the estimator having the least expected loss is usually deemed preferable.

Let p be an unknown parameter and $\hat{p}$ an estimate of p .The estimation error is the difference ($\hat{p}$ -p). The loss function is a function mapping estimation errors to the set of real numbers.

6.1 Squared Error Loss Function (SELF)

A commonly used loss function is the squared error loss function (SELF)

$$L(\hat{p}, p) = (\hat{p} - p)^2$$

The SELF is often used because it does not lead to extensive numerical computation.

Posterior Expected Loss $E(p_B - p)^2$ of Bayes estimators of the parameter p given θ for different priors under SELF for the proposed model are obtained by using direct method of integration and Lindley's approach and are given in Tables 4 and 5.

Prior	**Posterior Expected Loss of p_B under SELF**
Uniform	$p_B^2 + \frac{(n+1)(n+2)}{A^2} - 2p_B \frac{(n+1)}{A}$
Jeffrey's	$p_B^2 + \frac{(n+1)n}{A^2} - 2p_B \frac{n}{A}$
Exponential	$p_B^2 + \frac{(n+1)(n+2)}{(A+1)^2} - 2p_B \frac{(n+1)}{(A+1)}$
Mukherjee-Islam	$p_B^2 + \frac{(n+\alpha)(n+\alpha+1)}{A^2} - 2p_B \frac{(n+\alpha)}{A}$
Gamma	$p_B^2 + \frac{(n+\alpha)(n+\alpha+1)}{\left(A+\frac{1}{\sigma}\right)^2} - \frac{2p_B(n+\alpha)}{\left(A+\frac{1}{\sigma}\right)}$
Here $A = \sum_{i=1}^{n} \log_e \left(\frac{x_i}{\theta}\right)$	

Table 4: Posterior Expected Loss of Bayes Estimator of p under SELF for different priors (Direct Method)

Prior	Posterior Expected Loss of p_B under SELF
Uniform	$(p_B - p)^2 + \frac{p_B^2}{n} + \frac{2(p_B - p)p_B}{n}$
Jeffrey's	$(p_B - p)^2 + \frac{p_B^2}{n}$
Exponential	$(p_B - p)^2 + \frac{p_B^2}{n} - \frac{2(p_B - p)(p_B - 1)p_B}{n}$
Mukherjee-Islam	$(p_B - p)^2 + \frac{p_B^2}{n} + \frac{2(p_B - p)p_B\alpha}{n}$
Gamma	$(p_B - p)^2 + \left\{1 + 2(p_B - p)\left[\frac{\alpha - 1}{\hat{p}} - \frac{1}{\sigma}\right]\right\}\frac{p_B^2}{n} + \frac{2(p_B - p)p_B}{n}$

Table 5: Posterior Expected Loss of Bayes Estimator of p under SELF for different priors (Lindley's Approach)

6.2 The Precautionary Loss Function

Norstrom (1996) introduced an alternative asymmetric precautionary loss function, and also presented a general class of precautionary loss functions as a special case. These loss functions approach infinitely near the origin to prevent underestimation, thus giving conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function (APLF) is

$$L(\hat{p}, p) = \frac{(\hat{p} - p)^2}{\hat{p}}$$

Posterior Expected Loss $E\left(\frac{(\hat{p} - p)^2}{\hat{p}}\right)$ of Bayes estimators of the parameter p given θ for different priors under APLF for the proposed model are obtained by using direct method of integration and Lindley's approach and are given in Tables 6 and 7.

Prior	Posterior Expected Loss of p_B under APLF
Uniform	$p_B + \frac{(n+1)(n+2)}{A^2 p_B} - \frac{2(n+1)}{A}$
Jeffrey's	$p_B + \frac{n(n+1)}{A^2 p_B} - \frac{2n}{A}$
Exponential	$p_B + \frac{(n+1)(n+2)}{(A+1)^2 p_B} - \frac{2(n+1)}{(A+1)}$

Mukherjee-Islam	$p_B + \frac{(n+\alpha)(n+\alpha+1)}{A^2 p_B} - \frac{2(n+\alpha)}{A}$
Gamma	$p_B + \frac{(n+\alpha)(n+\alpha+1)}{p_B\left(A+\frac{1}{\sigma}\right)^2} - \frac{2(n+\alpha)}{\left(A+\frac{1}{\sigma}\right)}$
Here $A=\sum_{i=1}^{n}\log_e\left(\frac{x_i}{\theta}\right)$	

Table 6: Posterior Expected Loss of Bayes Estimator of p under APLF for different priors (Direct Method)

Priors	Posterior Expected Loss of p_B under APLF
Uniform	$\frac{(p_B-p)^2}{p_B} + \frac{p_B}{n}$
Jeffrey's	$\frac{(p_B-p)^2}{p_B} + \frac{(2p^2-p_B^2)}{np_B} + \frac{(p_B^2-p^2)}{np_B}$
Exponential	$\frac{(p_B-p)^2}{p_B} - \frac{(p_B^2-p^2)}{n} + \frac{p_B}{n}$
Mukherjee-Islam	$\frac{(p_B-p)^2}{p_B} + \frac{p^2+(p_B^2-p^2)(\alpha-1)}{np_B} + \frac{(p_B^2-p^2)}{np_B}$
Gamma	$\frac{(p_B-p)^2}{p_B} + \frac{1}{2}\left\{\frac{2p^2}{p_B^3} + 2\left(1-\frac{p^2}{p_B^2}\right)\left(\frac{\alpha-1}{p_B}-\frac{1}{\sigma}\right)\right\}\frac{p_B^2}{n} + \frac{(p_B^2-p^2)}{np_B}$

Table 7: Posterior Expected Loss of Bayes Estimator of p under APLF for different priors (Lindley's Approach)

7. Numerical Illustration

To illustrate the calculations of Bayes estimates of p given θ under different priors, we have generated a random sample of size 100 from Pareto type-I model (p=5, θ=2) with the help of Easy Fit Professional 5.5 software. The generated data are given below:

2.347741	2.215711	2.966724	2.49548
2.23219	2.474255	2.096693	2.417451
2.21434	2.317951	2.36612	2.209089
2.925098	3.454793	2.519692	2.558836
2.420827	2.005505	2.057287	2.546507
2.104143	2.222513	2.79783	2.345352

2.092396	2.312452	2.428589	2.929116
2.126033	3.191947	3.493986	2.496365
4.890878	3.099833	2.333158	2.143872
2.02951	3.631223	2.127282	3.808065
2.751409	2.025598	2.003405	2.269133
2.89437	2.337428	2.060617	4.387175
2.160667	2.438709	3.355947	2.258303
2.354059	2.636082	2.839061	3.169609
2.024442	2.603608	2.218725	2.504647
2.083845	2.086623	2.141131	2.068072
2.070496	2.357513	2.189409	2.121921
2.083327	2.245727	2.217183	2.274783
2.654614	2.933221	4.338843	2.219359
2.676281	2.415986	2.828695	2.022366
3.411296	2.263096	2.160593	2.111004
2.015468	2.37033	2.290202	2.682772
2.279763	2.313352	2.667013	3.081878
2.280491	2.840459	3.460497	2.421515
2.340969	3.956496	2.428091	2.66508

Bayes estimates of p, R(t) given θ= 2 are obtained under different priors using Lindley's Approach for the generated data and are given in Tables 8, whereas Bayes estimates of p (Direct method) and H(t) (Lindley's Approach) are given in Table 9.

Prior	p^B	$R(t)^B$ **For t=**				
		2	**3**	**4**	**5**	**6**
Uniform	4.5434	1.0000	0.1611	0.0450	0.0169	0.0076
Jeffrey's	4.4984	1.0000	0.1640	0.0463	0.0175	0.0080
Exponential	4.3410	1.0000	0.1743	0.0512	0.0199	0.0092
Mukh.-Islam α=1	4.5434	1.0000	0.1611	0.0450	0.0169	0.0076
α=2	4.5884	1.0000	0.1581	0.0436	0.0162	0.0073
α=3	4.6333	1.0000	0.1552	0.0422	0.0155	0.0069
Gamma $\alpha = 1, \sigma = 1$	4.3410	1.0000	0.1864	0.0603	0.0256	0.0128
$\alpha = 2, \sigma = 1$	4.3860	1.0000	0.1834	0.0597	0.0250	0.0125
$\alpha = 3, \sigma = 1$	4.4310	1.0000	0.1805	0.0576	0.0243	0.0121

Table-8: Bayes estimates of p and R(t) under different priors for θ=2

Table 8 reveals that Bayes estimator of p for $\theta = 2$ is quite close to its true value under Mukherjee-Islam's prior and it is approaching the true value as the value of α increases. The calculations for reliability and hazard rate functions have also been made in a similar manner at different values of t under different priors.

Prior	p^B	$H(t)^B$ **For t=**				
		2	**3**	**4**	**5**	**6**
Uniform	4.5434	2.2717	1.5144	1.1358	0.9086	0.7572
Jeffrey's	4.4984	2.2492	1.4994	1.1246	0.8996	0.7497
Exponential	4.3478	2.1705	1.4470	1.0852	0.8682	0.7235
Mukh.-Islam α=1	4.5434	2.2717	1.5144	1.1358	0.9086	0.7572
α=2	4.5884	2.2942	1.5294	1.1471	0.9176	0.7647
α=3	4.6333	2.3166	1.5444	1.1583	0.9266	0.7722
Gamma $\alpha = 1, \sigma = 1$	4.3478	2.1705	1.4470	1.0852	0.8682	0.7235
$\alpha = 2, \sigma = 1$	4.3908	2.1930	1.4620	1.0965	0.8772	0.7310
$\alpha = 3, \sigma = 1$	4.4339	2.2155	1.4770	1.1077	0.8862	0.7385

Table 9: Bayes estimates of H(t) under different priors for θ=2

8. Comparison and Conclusion

To compare the results numerically, we have calculated the values of Posterior expected loss under SELF and APLF by using the estimates of p by direct method and Lindley's approach under different priors. The calculations are shown in Table 10. It is revealed from the table that the results obtained by Asymmetric Precautionary Loss Function (APLF) are better than the Square Error Loss Function (SELF) under all the priors and the value of APLF is smallest under Gamma prior with α=1, σ=1 using direct method, whereas it is smallest under Uniform prior using Lindley's approach. Reliability and Hazard rate functions decrease as t increases, as revealed by Tables 8 and 9.

It can also be concluded from the above table that the results obtained by direct method are slightly better than Lindley's approach. Although, if in any case, it is almost impossible to solve the integration, Lindley's approach can be used to find out the approximate value of the estimates.

Prior	**Square Error Loss Function**		**Asymmetric Precautionary Loss Function**	
	Direct Method	**Lindley's Approach**	**Direct Method**	**Lindley's Approach**
Uniform	0.204383	0.373401	0.044980	0.091317
Jeffrey's	0.202359	0.453924	0.044980	0.111497
Exponential	0.187165	0.813787	0.043047	0.204983
Mukh.-Islam $\alpha=1$	0.204383	0.373401	0.044980	0.091317
$\alpha=2$	0.20843	0.304401	0.045875	0.074204
$\alpha=3$	0.216525	0.247176	0.047656	0.067719
Gamma $\alpha = 1, \sigma = 1$	0.187165	0.813787	0.043040	0.581880
$\alpha = 2, \sigma = 1$	0.190872	0.697816	0.043900	0.515170
$\alpha = 3, \sigma = 1$	0.198284	0.642643	0.045600	0.450860

Table 10: The values of SELF and APLF under different priors using direct method and Lindley's approach

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