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A COMPOUND OF ZERO TRUNCATED GENERALIZED NEGATIVE BINOMIAL DISTRIBUTION WITH GENERALIZED BETA DISTRIBUTION

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Abstract

In this paper, we provide a chronological overview of the recent developments in the compounding of distributions. An attempt has been made to obtain a compound of zero truncated generalized negative binomial distribution (Z_{tgnbd}) with that of Generalized Beta Distribution (GBD). Factorial and ordinary crude moments of some zero truncated compound distributions have also been discussed. The compound is then specialized by varying the value of β in ((Z_{tgnbd})).

Key Words: Zero truncated generalized negative binomial distribution, Generalized beta distribution, Compound distributions, Factorial and crude moments.

1. Introduction

As far as the problem of the compounding of the probability distributions is concerned the work has been done in this area since 1920. It is well known that the parameter in Poison distribution was considered to be a gamma variate in the famous paper by Greenwood and Yule (1920). The interrelationships among compound and generalized distributions were Ist explored by Gurland (1957).Some important remarks on mixtures of distributions were discussed by Molenaar (1965).

Starting in the early 1970s, Dubey (1970) derived a compound gamma, beta and F distribution by compounding a gamma distribution with another gamma distribution and reduced it to the beta Ist and 2nd kind and to the F distribution by suitable transformations. For a review of the literature one may refer to Johnson et al. (1992) and Balakrishanan (1994).The problem of compounding of distributions was further addressed by Gerstenkorn who proposed several compound distributions, (1993) he obtained compound of gamma distribution with exponential distribution by treating the parameter of gamma distribution as an exponential variate and (1996) he also obtained compound of polya with beta. Gerstenkorn (2004) also obtained a compound of generalized negative binomial distribution with generalized beta distribution by treating the parameter of generalized negative binomial distribution as generalized beta distribution. Cekanavicius and Vellaisamy (2010) used Compound Poisson distributions and Signed compound Poisson measures for approximation of the Markov binomial distribution. Most recently Sajid Ali, Mohammad Aslaman and Syed Mohsin Ali (2011) improved the informative prior for the mixture of Laplace distribution under different loss functions.

2. The compounding of probability distributions

We begin by giving a definition and the relations needed for the compounding of distributions.

Let Xy be a random variable with a distribution function $F(x|y)$ that depends on parameter y. Suppose parameter y is a random variable Y with distribution function $G(y)$. Then the distribution that has the distribution function of X defined by the formula

$$H(x) = \int_{-\infty}^{\infty} F(x \mid cy)\, dG(y) \tag{2.1}$$

will be called compound, where c is an arbitrary constant or a constant bounded on some interval (Gurland (1957)).
The occurrence of the constant c in (2.1) has a practical justification in as much as the distribution of a random variable, in describing a phenomenon, often depends on a parameter that is itself a realization of another random variable multiplied by a certain constant. The variable that has distribution function (2.1) will be symbolized by "$X \wedge Y$" and will be called a compound of the variable X with respect to the "compounding" Y.
Relation (2.1) will be symbolized as follows:

$$H(x) \equiv F(x \mid cy) \underset{Y}{\wedge} G(y).$$

Consider the case when one variable is discrete, with probability function $P\left(X = x_i \middle| cy\right)$ where parameter y is a random variable Y with density $g(y)$. Then (2.1) is expressed by the formula

$$h(x_i) = P(X = x_i) = \int_{-\infty}^{\infty} g(y) P(X = x_i \mid cy)\, dy \tag{2.2}$$

3. The compounding of the zero truncated generalized negative binomial distribution with the generalized beta distribution

Zero truncated generalized negative binomial distribution is a distribution with the probability function given by the formula (see Famoye and Consul (1992))

$$Z_{tgnbd}\,(x;n,p,\beta) = P_\beta\,(x,n,p) = \frac{\dfrac{n}{n+\beta x}\dbinom{n+\beta x}{x} p^x (1-p)^{n+\beta x - x}}{1-(1-p)^n},\ x = 1, 2 \ldots. \tag{3.1}$$

where

$$0 \le p < 1,\ n > 0,\quad \beta p < 1,\ \beta \ge 1 \tag{i}$$

$$0 \le p \le 1,\ n \in N,\ \beta = 0 \qquad \text{(ii)}$$

For β =1 one obtains from (3.1) zero truncated negative binomial distribution. If $n \in N$, for β=0 one obtains from (3.1) zero truncated binomial distribution and for $\beta=1$, zero truncated Pascal distribution (in accordance with N. Johnson, S. Kotz and A. Kemp (1992)).

By the generalized beta distribution we mean a distribution given by the density function

$$GB(y;a,b,w,r) = \begin{cases} \dfrac{ay^{r-1}}{(bw)^{r/a} B(r/a,w)} \left(1 - \dfrac{y^a}{bw}\right)^{w-1} & for\ \ 0 < y < (bw)^{1/a} \\ 0 & for\ \ y \le 0 \text{ or } y \ge (bw)^{1/a} \end{cases} \qquad (3.2)$$

where $a, b, r, w > 0$ and $B(r/a, w)$ is a beta function

Distribution (3.2) is a special limit case of the Bessel distribution investigated by T. Srodka (1973). It was also analyzed by J. Seweryn (1986) and by W. Oginiski (1979) was applied in reliability theory.

Let us consider Z_{tgnbd} (3.1) that depends on cy:

$$P_\beta\left(x;n,cy\right) = \frac{\dfrac{n}{n+\beta x}\dbinom{n+\beta x}{x}\left(cy\right)^x\left(1-cy\right)^{n+\beta x - x}}{1-\left(1-cy\right)^n},\ \ x = 1,2,..., \qquad (3.3)$$

where $0 < cy < 1$, $n = 1,2,\ldots$, $\beta cy < 1$, $\beta \ge 1$, and Y is a random variable with GBD (3.2)

Theorem 3.1: The probability function of the compound distribution

$$Z_{tgnbd} \underset{Y}{\wedge} GBD$$

is given by the formula

$$P_\beta GB\left(x\right) = D_1 \sum_{s=1}^{\infty}\sum_{k=0}^{\infty}\left(-c\right)^k \binom{ns+\beta x - x}{k}\left(bw\right)^{k/a} B\left(\frac{x+r+k}{a}, w\right) \qquad (3.1.1)$$

where $$D_1 = \frac{\frac{n}{n+\beta x}\binom{n+\beta x}{x} c^x (bw)^{x/a}}{B(r/a, w)}$$

$x = 1,2,\ldots$, $a, b, w, r > 0$, where $\beta cy < 1$, $0 < cy < 1$, $n > 0$, $\beta \geq 1$.

Proof: From formulas (2.2), (3.2) and (3.3) we have

$$P_\beta GB(x) = a\ D_2 \int_0^{(bw)^{1/a}} y^{x+r-1}\left(1-\frac{y^a}{bw}\right)^{w-1} \sum_{S=1}^{\infty}(1-cy)^{ns+\beta x-x} dy$$

$$= aD_2 \sum_{s=1}^{\infty}\sum_{k=0}^{\infty}(-c)^k \binom{ns+\beta x - x}{k} \int_0^{(bw)^{1/a}} y^{x+r+k-1}\left(1-\frac{y^a}{bw}\right)^{w-1} dy$$

Where, $$D_2 = \frac{\frac{n}{n+\beta x}\binom{n+\beta x}{x} c^x}{(bw)^{r/a} B(r/a, w)}$$

Substituting, $y^a/bw = t$, we get

$$P_\beta GB(x) = D_1 \sum_{s=1}^{\infty}\sum_{k=0}^{\infty}(-c)^k \binom{ns+\beta x - x}{k} (bw)^{k/a} \int_0^1 t^{\frac{x+r+k}{a}-1}(1-t)^{w-1} dt$$

$x = 1,2,\ldots$, $a, b, w, r > 0$, $n > 0$, $\beta \geq 1$, $0 < c \leq \frac{1}{\beta (bw)^{1/a}}$

and from the definition of the beta function, we obtain (3.1.1). In special case when n, $\beta \in N$ we have a simpler formula given by

$$P_\beta GB(x) = D_1 \sum_{s=1}^{\infty}\sum_{k=0}^{ns+\beta x - x}(-c)^k \binom{ns+\beta x - x}{k} (bw)^{k/a} B\left(\frac{x+r+k}{a}, w\right) \tag{3.1.2}$$

Case (ii) of (3.1) i.e. the case when $\beta = 0$, is discussed in next section 3.2

3.2 Special Cases

In the case when β = 0 (zero truncated binomial distribution) the proof of theorem (3.1) i.e. similar to but even simpler than, the proof given above. More specifically, we get

$$P_\beta GB(x) = D_1^* \sum_{s=1}^{\infty} \sum_{k=0}^{ns-x} \binom{ns-x}{k} (-c)^k (bw)^{k/a} B\left(\frac{x+r+k}{a}, w\right)$$

Where $$D_1^* = \frac{\binom{n}{x} c^x (bw)^{x/a}}{B(r/a, w)}$$

Here we get a compound of zero truncated binomial distribution with the generalized beta distribution:

zero truncated binomial $\underset{Y}{\wedge}$ generalized beta (β = 0)

$$P_0 GB(x) = \frac{\binom{n}{x} c^x (bw)^{x/a}}{B(r/a, w)} \sum_{s=1}^{\infty} \sum_{k=0}^{ns-x} (-c)^k \binom{ns-x}{k} (bw)^{k/a} B\left(\frac{x+r+k}{a}, w\right) \qquad (3.2.1)$$

$$x = 1, 2, \ldots, n$$

From (3.1.2) we also have the compound of zero truncated generalized negative binomial with the beta distribution:

Z_{tgnbd} $\underset{Y}{\wedge}$ beta (b=1/w, a=1)

$$P_\beta B(x) = \frac{\frac{n}{n+\beta x} c^k \binom{n+\beta x}{x}}{B(r, w)} \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{ns+\beta x-x}{k} B(x+r+k, w)$$

It can be demonstrated that

$$\sum_{k=0}^{\infty} (-1)^k \binom{ns+\beta x-x}{k} B(x+r+k, w) = B(x+r,\ w+ns+\beta x-x)$$

holds. Hence, we get

$$P_\beta B(x)=\frac{n \quad c^k}{n+\beta x}\binom{n+\beta x}{x}\sum_{s=1}^{\infty}\frac{B\left(x+r,w+ns+\beta x-x\right)}{B\left(r,w\right)}$$

a results that for $\beta = 1$ and $c = 1$, yields zero truncated negative binomial $\underset{Y}{\wedge}$ beta

$$P_1 B(x)=\binom{n+x-1}{x}\sum_{s=1}^{\infty}\frac{B\left(x+r,w+ns\right)}{B\left(r,w\right)}$$

Further, a consequence of theorem 3.1 is for β = 1, n = 1,2,…,, the compound distribution zero ntruncated Pascal binomial $\underset{Y}{\wedge}$ generalized beta

$$P_1 GB(x)=\binom{n+x-1}{x}\frac{c^x\left(bw\right)^{x/a}}{B\left(r/a,w\right)}\sum_{s=1}^{\infty}\sum_{k=0}^{ns}\left(-c\right)^k\binom{ns}{k}\left(bw\right)^{k/a}B\left(\frac{x+r+k}{a},w\right)$$

(3.2.2)

4. Factorial moments and ordinary (crude) moments of a compound of zero truncated negative binomial distribution with the generalized beta distribution.

Let X_y and X be a random variable with distribution functions $F(x/y)$ and $H(x)$, respectively (2.1), and let parameter y have distribution $G(y)$. Then, when one keeps in mind the formula for the so-called factorial polynomial

$x^{[l]} = x(x-1)(x-2)\dots(x-(l-1))$,

$$m_{[l]}=E\left(X^{[l]}\right)=\int_{\infty}^{\infty}E\left(X_y^{[l]}\right)dG(y) \tag{4.1}$$

is called a factorial moment of order l of the variable X with compound distribution (2.1)

Relation (4.1) will be symbolized as follows:

$$E\left(X_y^{[l]}\right)\underset{Y}{\wedge}G\left(y\right). \tag{4.2}$$

Theorem 4.1 The factorial moment of order l of the compound distribution zero truncated negative binomial $\underset{Y}{\wedge}$ generalized beta is given by the formula

$$m_{[l]}^{Z_{tnb}-GB} = \frac{n^{[l-1]} c^l (bw)^{l/a}}{B(r/a,w)} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \binom{ns-l}{k} (-c)^k (bw)^{/a} B\left(\frac{l+r+k}{a}, w\right) \quad (4.1.1)$$

Proof The factorial moment of order l of zero truncated negative binomial distribution is given by the formula

$$m_{[l]}^{Z_{tnb}-GB}(n,p) = \frac{\left(\frac{p}{q}\right)^l n^{[l-1]}}{1-(1-p)^n}, 0 < p < 1,\ q = 1-p$$

Consequently, by (4.2), the factorial moment of this order of distribution (3.2.2) is, if we let $p=cy$, the following:

$$m_{[l]}^{Z_{tnb}-GB} = m_{[l]}^{z_{tnb}}(n,cy) \underset{Y}{\wedge} GB(y;a,b,r,w)$$

$$= \frac{a \quad n^{[l,-1]}}{(bw)^{r/a} \, B(r/a,w)} c^l \int_0^{(bw)^{l/a}} y^{l+r-1} \left(1-\frac{y^a}{bw}\right)^{w-1} \sum_{s=0}^{\infty} (1-cy)^{ns-l} \, dy$$

$$= \frac{a \; n^{[l,-1]} c^l}{(bw)^{r/a} \, B(r/a,w)} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \binom{ns-l}{k} (-c)^k \int_0^{(bw)^{1/a}} y^{l+r+k-1} \left(1-\frac{y^a}{bw}\right)^{w-1} dy$$

Substituting, $\frac{y^a}{bw} = t$, we get

$$m_{[l]}^{Z_{tnb}-GB} = \frac{n^{[l-1]} c^l (bw)^{l/a}}{B(r/a,w)} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \binom{ns-l}{k} (-c)^k (bw)^{/a} \int_0^1 t^{\frac{l+r+k}{a}} (1-t)^{w-1} dt$$

a result that yields (4.1.1)

A special case of theorem 4.1 is the following theorem.

Theorem 4.2: The factorial moment of order l of the compound distribution zero truncated negative binomial $\underset{y}{\wedge}$ beta (b=1/w, a=1, c=1) is given by the formula (for w>1)

$$m_{[l]}^{Z_{tnb}-B} = \frac{n^{[l-1]}}{B(r,w)} \sum_{s=0}^{\infty} B(l+r, w+ns-l) \tag{4.2.1}$$

Proof: In this case we have

$$m_{[l]}^{Z_{tnb}-B} = \frac{n^{[l-1]}}{B(r,w)} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \binom{ns-l}{k} (-1)^k B(l+r+k, w)$$

It can be demonstrated that

$$\sum_{K=0}^{\infty} \binom{ns-l}{k} (-1)^k B\left(l+r+k, w\right) = B\left(l+r, w+ns-l\right)$$

Thus giving formula (4.2.1)

The ordinary (crude) moments of the compound distribution under consideration are obtained by using the formula

$$m_l = \sum_{K=0}^{l} S_k^l m_{[k]},$$

Where S_k^l stands for the so-called stirling numbers of the second kind. Bohlmann in 1913 seems to be the first to give this formula; the tables for these numbers can be found for instance, in A. Kaufmann in 1968 or in J. Lukasiewicz and M. Warmus in 1956.

References

1. Ali, sajad, Aslam, Mohammed and Kazmi, Ali Mohsin (2011). Improved informative prior for the mixture of Laplace distribution under different loss functions, Journal of Reliability and Statistical studies, 4(2), p. 57-82
2. Bohlmann, G. (1913). Formulierung und Begrundung zweier Hilfssatze der mathematischen Statistik, 14, p. 341-409.
3. Cekanavicius, V. and Vellaisamy, P. (2010). Compound poisson and signed compound Poisson approximations to the Markov binomial law, 16, p. 1114-1136
4. Dubey, D.S. (1970). Compound gamma, beta and F distributions, Metrika., 16(1), p. 27-31.
5. Famoye, F. and Consul, P.C. (1992). The truncated generalized negative binomial distribution, Journal of Applied Statistical Sciences, p. 141-157.
6. Gerstenkorn, T. (2004). A compound of the generalized negative binomial distribution with the generalized beta one, CEJM, 2(4), p. 527-537.

7. Gerstenkorn, T. (1993). A compound of the generalized gamma distribution with the exponential one, Bull. Soc. Sci. Letters Lodz, Vol. 43(1), Recherches surles deformations., 16(1), p. 5-10.
8. Gerstenkorn, T. (1996). A compound of the Polya distribution with the beta one, Random Oper. and Stoch. Equ., 4(2), p. 103-110.
9. Greenwood, M. and Yule, G.U. (1920). An inquiry into the nature of frequency distribution representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents, J. Roy. Stat. Soc., 83, p. 255-279.
10. Gurland, J. (1957). Some interrelations among compound and generalized distributions, Biometrika, 44, p. 265-268.
11. Jakuszenkow, H. (1973). Nowe zlozenia rozkladow (New compounds of distributions;in Polish), Przeglad Statystyczny, 20(1), p. 67-73.
12. Johnson, N.L., kotz, S. and Balakrishanan, N. (1994),Continuous Univariate Distributiions.,John Willy& Sons, inc, New York,.
13. Johnson, N.L., Kotz, S. and Kemp, A. (1992). Univariate Discrete Distributions, 2nd Ed., J. Wiley, New York.
14. Kaufmann, A. (1968). Introduction a la Combnatorique en Vue des Applications, Dunold, Paris
15. Lukaszewicz, J and Warmus, W(1956). Metody Numertczne i Grficzne (Numerical and Graphical Methods;in Polish), PWN, Warszawa.
16. Molenaar, W. (1965). Some remarks on mixtures of distributions, Bull. Intern.Statist. Institute; 35th Session of the Intern. Statist. Inst., Beograd., 2, p. 764-765.
17. Oginski, L. (1979). Application of a distribution of the Bessel type in reliability theory, Polish,Matematkya, 12, p. 31-42.
18. Seweryn, J.G. (1986). Some probabilistic properties of Bessel distribution, Matematkya, 19, p. 69-87.
19. Srodka, T. (1973). On some generalized Bessel- type probability distribution, Matematkya, 4, p. 5-31.