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BAYES PRE-TEST ESTIMATION OF SCALE PARAMETER OF WEIBULL DISTRIBUTION UNDER DIFFERENT LOSS FUNCTIONS USING PROGRESSIVE TYPE II CENSORED SAMPLE

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Abstract

In this paper, we propose Bayes Shrinkage (BS) estimators and Bayes Pre-test (BP) estimators for the scale parameter of Weibull distribution using type II progressive censored sample and study their properties under Squared Error Loss Function (SELF) and LINEX loss function (LLF). The results show that the suggested BP estimators, in terms relative risk with respect to both SELF and LINEX loss functions, have better performance than the BS estimators.

Key words: Bayes pre-test estimator, Progressive type II censored sample, Squared error loss function, LINEX loss function, Weibull distribution.

2010 Mathematics Subject Classification: 62J07, 62F15, 62N01.

1. Introduction

The Weibull model is used widely in reliability and life testing. Weibull (1951) showed that the distribution is useful to describe the wear-out or fatigue failures. Lieblein and Zelen (1956) used it as model in the study of diameter of ball bearings. It is also used as model for vacuum tube (see Kao (1959)). Tadikamalla (1978) used it in inventory control. Mittnik and Rachev (1993) found that the Weibull distribution may be used as a statistical model for stock returns. Apart from these interesting applications, there are plenty of other papers describing the application of Weibull distribution in different scientific studies. Looking at its wide varieties of applications, many people have carried theoretical studies on different forms of Weibull distribution under various sampling schemes. However, the use progressive sampling scheme has not attracted many researchers though the scheme has its prime importance in reliability and life testing experiments. The toughness in mathematical tractability of the estimators and tests may be one of the reasons for not attracting the researcher.

In the present study we concentrate on obtaining Bayes Shrinkage (BS) estimators and Bayes Pre-test (BP) estimators for the scale parameter of Weibull distribution using type II progressive censored sample by considering the form of density as given below

$$f(x;p,\theta)=\frac{px^{p-1}}{\theta}e^{-\frac{x^p}{\theta}} \quad , \quad x\ge 0,\, p,\theta>0, \tag{1}$$

where p is the shape parameter and θ is the scale parameter.

In many situations we have prior information about the unknown parameter θ as a guess value, say θ_0, which is desirable to be incorporated in the estimation. For the first time, Thompson (1968) suggested a new type of estimation scheme called shrinkage estimation, for unknown parameter θ when a guess value θ_0 is available. Normally, shrinkage estimators perform better than the usual estimators when the guess value is close to the true value of the parameter. Exploiting this advantage, in statistics literature, many research papers have been written on obtaining shrinkage estimators for different parameters or parametric functions, under different life of distributions, using the data under different life testing schemes. The few interesting references are Pandey (1983), Pandey and Upadhyay (1985) and Pandey et al. (1989). The research papers by Singh et al. (2008), Prakash and Singh (2006,2009 and 2010) are recent advances in shrinkage estimation as they deal with estimation of scale parameter of Weibull distribution using complete and usual censored data under different loss functions. In the present research work, on the line of these papers, we have obtained shrinkage estimators of scale parameters of Weibull distribution using progressive type II censored data and studied their properties under squared error and LINEX loss functions.

In progressive censoring an experimenter desires to remove units at points other than the final termination point that is, (see Balakrishnan and Aggarwala(2000)) after observing the first failure, R_1 units are randomly selected and removed ; after observing the second failure,R_2 units are randomly selected and removed; and likewise when the i-th failure units is observed R_i units are randomly selected and removed ;i=3, 4,… ,m. The experiment terminates when the m-th failure is observed and the remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ units are removed. This type censoring scheme is generally referred as type II progressive censored scheme. The complete sample and usual censored sample are special cases of type II progressive censored scheme for $R_1=R_2= \ldots = R_m=0$ and for $R_1=R_2= \ldots = R_{m-1}=0$, $R_m= n-m$, respectively. Balakrishnan (2007) gave a detailed study on the recent developments in inferential aspects based on type II progressive censored scheme. For some of the earlier research, under progressive censored data, we refer to Balakrishnan et al. (2003), Fernandez (2004), Guilbaud (2004), Soliman (2008), Balakrishnan and Cramer (2008), Balakrishnan and Dembinska (2008), Pradhan and Kundu (2009) and Al-Aboud (2009).

Now, let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ be Type II progressive censored sample from the Weibull distribution as defined in (1). Then the joint density of progressive censored sample is

$$f(x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}) = C\prod_{i=1}^{m} f(x_{i:m:n})(1-F(x_{i:m:n}))^{R_i}, \qquad 0 \le x_{1:m:n} \le x_{2:m:n} \le \ldots \le x_{m:m:n} \tag{2}$$

where $C= n(n-R_1-1)(n-R_1-R_2-2) \ldots (n-R_1-R_2- \ldots - R_{m-1}-m+1)$.

Since $X_{1:m:n}$, $X_{2:m:n}$, … , $X_{m:m:n}$ are from Weibull distribution, the joint density is

$$f(x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n};\theta) = C\prod_{i=1}^{m}\left(\frac{p}{\theta}\right)^{m} x_{i:m:n}^{p-1}\exp\left(-\frac{\sum_{i=1}^{m}(R_i+1)x_{i:m:n}^{p}}{\theta}\right),$$

$$0 \le x_{1:m:n} \le x_{2:m:n} \le \dots \le x_{m:m:n} \tag{3}$$

In this paper, we assume that the shape parameter is known. Then the ML estimator for the scale parameter θ is

$$T = \frac{\sum_{i=1}^{m}(R_i+1)X_{i:m:n}^{p}}{m} . \tag{4}$$

Now, let us make the transformation $Y_{i:m:n} = X_{i:m:n}^{p}$ so that $Y_{1:m:n}, Y_{2:m:n}, \dots, Y_{m:m:n}$ is a progressive censored sample from exponential distribution with mean θ. Then using the results of Balakrishnan and Aggarwala (2000) we can show that $\frac{2mT}{\theta}$ follows χ^2_{2m} distribution. Consequently, T has Gamma distribution with parameters m and θ/m .The probability density function is

$$f(t;\theta,m) = \frac{t^{m-1}\left(\frac{m}{\theta}\right)^{m} e^{-\frac{mt}{\theta}}}{\Gamma(m)} . \qquad t>0 \tag{5}$$

2. Class of Maximum Likelihood (ML) Estimators

We know that T is the ML estimator θ. Using this fact we define a class of estimators of the scale parameter θ as

$$D = \gamma\, T, \quad \gamma \in R^{+} . \tag{6}$$

The risk of the estimator D under squared error loss function is given as

$$R_{SELF}(D) = \theta^2\left(\gamma^2\frac{m+1}{m} - 2\gamma + 1\right) . \tag{7}$$

Now, the constant γ which minimizes $R_{SELF}(D)$ is $\lambda_1 = \frac{m}{m+1}$.

In many real life situations, errors are not symmetric around the true value. In such cases, instead of using SELF, one can advantageously use LINEX loss function (asymmetric loss function) introduced by Varian (1975) which is given below;

$$L(\Delta) = b(e^{a\Delta} - a\Delta - 1) \quad , \tag{8}$$

where $\Delta = \frac{\hat{\theta}}{\theta} - 1$, ‘b’ is the scale parameter and ‘a’ is the shape parameter. The sign and value of ‘a’ represents the direction and degree of asymmetry. The positive value of ‘a’ is used when over estimation is more serious than under estimation and negative value is used for the other case. This property of the LINEX loss function has been considered by many researchers. The few important articles using the loss function are Varian (1975) , Basu and Ebrahimi(1991), Srivastava and Tanna(2001), and Srivastava and Shah (2010).

Let b=1 and $m > a\gamma$

Now, the risk of the estimator D under LINEX loss function is

$$R_{LLF}(D) = e^{-a}\left(1 - \frac{a\gamma}{m}\right)^{-m} + a(1-\gamma) - 1 \ . \tag{9}$$

The constant γ which minimizes $R_{LLF}(D)$ is given as

$$\gamma_2 = \frac{m}{a}\left(1 - e^{-\frac{a}{m+1}}\right).$$

Based on constants γ_1 and γ_2 we define two estimators as

$$D_1 = \gamma_1 T,$$

and

$$D_2 = \gamma_2 T.$$

On substitution of the values of γ_1 and γ_2 in (7) and (9) respectively, we get the respective risks of two estimators under square error loss function and LINEX loss function. Further we obtain risk of one estimator using the loss function which is being used to obtain the other estimator by minimizing the corresponding risk.

$$R_{SELF}(D_1) = \frac{\theta^2}{m+1} \quad ,$$

$$R_{LLF}(D_1) = e^{-a}\left(1 - \frac{a\gamma_1}{m}\right)^{-m} - a(1-\gamma_1) - 1,$$

$$R_{SELF}(D_2) = \theta^2\left(\gamma_2 \frac{m+1}{m} - 2\gamma_2 + 1\right) \quad ,$$

$$R_{LLF}(D_2) = (1+m)\left(e^{-\frac{a}{m+1}} - 1\right) + a \quad .$$

3. Bayes pre-test estimators

Instead of using maximum likelihood estimator, one can use Bayes estimator for obtaining shrinkage estimators. This idea prompted us to consider following prior distribution to obtain Bayes estimator of the parameter of the distribution.

$$g(\theta)=\frac{e^{-\frac{c}{\theta}}}{\theta^{b}} \qquad .\theta>0 \tag{10}$$

Then the posterior distribution is given by

$$\prod(\theta|x_{1:m::n},x_{2:m:n},\dots,x_{m:m:n})=\frac{(mt+c)^{m+b-1}\exp\left(-\frac{mt+c}{\theta}\right)}{\theta^{m+b}\Gamma(m+b-1)} \quad ,$$

and the corresponding Bayes estimator under the SELF is given by

$$\hat{\theta}_B=\frac{mT+c}{m+b-2} \tag{11}$$

For utilizing the prior information (guess estimator) we choose the values of b and c such that the expectation of Bayes estimator is equal to guess estimator θ_0. That is

$$E(\hat{\theta}_B)=\theta_0 \quad .$$

This gives us $c=\theta_0(q-m)$ where q = m+b-2. Substituting this value of c in (11), we get

$$\bar{\theta}_B=k\,T+(1-k)\,\theta_0 \tag{12}$$

where $k=\frac{m}{q}$, the estimator $\bar{\theta}_B$ is Bayes shrinkage estimator. Using the estimator given in (12), with the use of preliminary testing, we suggest the following two pre-test estimators.

$$\tilde{\theta}_{BS}=\begin{cases}k\,T+(1-k)\,\theta_0 & r_1<T<r_2\\ D_1 & \text{otherwise}\end{cases}, \tag{13}$$

$$\tilde{\theta}_{BL}=\begin{cases}k\,T+(1-k)\,\theta_0 & r_1<T<r_2\\ D_2 & \text{otherwise}\end{cases}, \tag{14}$$

where r_1 and r_2 are boundaries of the acceptance region of a test of the hypothesis $H_0:\theta=\theta_0$ against the alternative H_1: $\theta\neq\theta_0$. Define $r_1=\frac{\theta_0\chi_1^2}{2m}$ and $r_2=\frac{\theta_0\chi_2^2}{2m}$, where χ_1^2 and χ_2^2 are respectively lower and upper αth percentile values of chi-square distribution with 2m degrees of freedom. In the next section we derive risks of the above two estimators under SELF and LLF.

4. Risk of pre-test estimators

The risk of the estimator $\tilde{\theta}_{BS}$ under SELF is defined as follows:

$$R_{SELF}(\tilde{\theta}_{BS}) = E(\tilde{\theta}_{BS} - \theta)^2 = \int_{r_1}^{r_2} (k(t-\theta)-(\theta-\theta_0))^2 f(t)\,dt$$
$$+\int_0^{\infty} (\gamma_1 t-\theta)^2 f(t)\,dt - \int_{r_1}^{r_2} (\gamma_1 t-\theta)^2 f(t)\,dt$$

by using the transformation $x = \frac{mt}{\theta}$ and evaluating the integral we get

$$R_{SELF}(\tilde{\theta}_{BS}) = \theta^2 \left(\begin{array}{l} k^2 \left(\frac{m+1}{m}[\, I(r_2', m+2) - I(r_1', m+2)\,] - 2\lambda\, [\, I(r_2', m+1) - I(r_1', m+1)\,] + \lambda^2\, [\, I(r_2', m) - I(r_1', m)\,] \right) \\ -2k(1-\lambda)\left([\, I(r_2', m+1) - I(r_1', m+1)\,] - \lambda\, [\, I(r_2', m) - I(r_1', m)\,] \right) + (1-\lambda)^2\, [\, I(r_2', m) - I(r_1', m)\,] \\ +[\, \gamma_1 \frac{m+1}{m} - 2\gamma_1 + 1] - \gamma_1^2 \frac{m+1}{m}\, [\, I(r_2', m+2) - I(r_1', m+2)\,] + 2\gamma_1\, [\, I(r_2', m+1) - I(r_1', m+1)\,] \\ -[\, I(r_2', m) - I(r_1', m)\,] \end{array} \right)$$

, (15)

where $r_1' = \frac{\lambda\,\chi_1^2}{2}$, $r_2' = \frac{\lambda\chi_2^2}{2}$, $\lambda = \frac{\theta_0}{\theta}$ and I(x, n) is the cdf of the gamma distribution given by $I(x,n) = \frac{\int_0^x t^{n-1}\, e^{-t} dt}{\Gamma(n)}$. The risk function of $\tilde{\theta}_{BL}$ under square loss function obtained from (15)

by replacement γ_2 instead of γ_1. Now, we can find risk equation of estimator $\tilde{\theta}_{BS}$ under LINEX loss function as follows:

$$R_{LLF}(\tilde{\theta}_{BS}) = E(\tilde{\theta}_{BS} | L(\Delta)) = \int_{r_1}^{r_2} (e^{a(\frac{k\,t+(1-k)\,\theta_0}{\theta}-1)} - a_1(\frac{kt+(1-k)\theta_0}{\theta}-1)-1) f(t)dt\,,$$
$$+\int_0^{\infty} (e^{a(\frac{\gamma_1 t}{\theta}-1)} - a(\frac{\gamma_1 t}{\theta}-1)-1) f(t)dt - \int_{r_1}^{r_2} (e^{a(\frac{\gamma_1 t}{\theta}-1)} - a(\frac{\gamma_1 t}{\theta}-1)-1) f(t)dt$$

Also by using the transformation $x = \frac{mt}{\theta}$ and evaluating the integral we get

$$R_{LLF}(\tilde{\theta}_{BS}) = \left(\frac{\exp(a((1-\lambda)k-1)}{(1-\frac{ak}{m})^m} \right) [I(r_2',m)-I(r_1',m)] - a((1-\lambda)k-1))[I(r_2',m)-I(r_1',m)]$$
$$-ak[I(r_2',m+1)-I(r_1',m+1)] + \frac{e^a}{(1-\frac{a\gamma_1}{m})^m} - a(\gamma_1-1)-1-\frac{e^a}{(1-\frac{a\gamma_1}{m})^m}[I(r_2',m)-I(r_1',m)]$$
$$+a\gamma_1[I(r_2',m+1)-I(r_1',m+1)] - a[I(r_2',m)-I(r_1',m)] \quad (16)$$

The risk function of $\tilde{\theta}_{BL}$ under LNIEX loss function can be obtained from (16) by replacing γ_1 by γ_2.

5. Relative Risk

To study the properties of estimators $\tilde{\theta}_{BS}$, $\tilde{\theta}_{BL}$ under SELF and LLF we compare the relative risks of the estimators given above. The relative risk of $\tilde{\theta}_{BS}$ with respect to D_1 and $\tilde{\theta}_{BL}$ with respect to D_2 under SELF and LLF are

$$RR_{SELF}(\tilde{\theta}_{BS}, D_1) = \frac{R_{SELF}(D_1)}{R_{SELF}(\tilde{\theta}_{BS})}, \quad (17)$$

$$RR_{SELF}(\tilde{\theta}_{BL}, D_2) = \frac{R_{SELF}(D_2)}{R_{SELF}(\tilde{\theta}_{BL})}, \quad (18)$$

$$RR_{SELF}(\tilde{\theta}_{BL}, D_1) = \frac{R_{SELF}(D_1)}{R_{SELF}(\tilde{\theta}_{BL})}, \quad (19)$$

$$RR_{LLF}(\tilde{\theta}_{BS}, D_1) = \frac{R_{LLF}(D_1)}{R_{LLF}(\tilde{\theta}_{BS})}, \quad (20)$$

$$RR_{LLF}(\tilde{\theta}_{BL}, D_2) = \frac{R_{LLF}(D_2)}{R_{LLF}(\tilde{\theta}_{BL})}. \quad (21)$$

and

$$RR_{LLF}(\tilde{\theta}_{BS}, D_2) = \frac{R_{LLF}(D_2)}{R_{LLF}(\tilde{\theta}_{BS})} \quad (22)$$

We observe that the equations for relative risks depend on m, b, a, α and λ .To demonstrate the performance of the proposed estimators under SELF and LLF, we have considered few values of the constants as: m = 6, 9, 12, b= 2, 4, 8, a=0.5, 1.0, α=0.01, 0.05, λ= 0.25(0.25)1.75.The Tables 1-4 give the values of the relative risks for the above given values of constants. Based these tables we have the following conclusions.

5. Conclusion

(i) The relative risks of both the estimators $\tilde{\theta}_{BS}$ and $\tilde{\theta}_{BL}$, with respect to the estimators D_1 and D_2 respectively under SELF and LLF, are increasing functions of b provided $0.75 \le \lambda \le 1.25$. Further the relative risk of the estimator $\tilde{\theta}_{BS}$ with respect D_1 is a decreasing function of m (this results in saving of sample units) whereas the relative risk of estimator $\tilde{\theta}_{BL}$ with respect D_2 is increasing in m.

(ii) It is observed that the relative risk of both the estimators $\tilde{\theta}_{BS}$ and $\tilde{\theta}_{BL}$ under SELF and LLF are decreasing functions of α.

(iii) The estimator $\tilde{\theta}_{BL}$ has higher relative risk with respect to D_2 under SELF and LLF than the relative risk of the estimator $\tilde{\theta}_{BS}$ with respect to D_1.

(iv) The highest relative risk of the estimator $\tilde{\theta}_{BL}$ with respect to D_2 under SELF when a=0.5.

(v) The relative risk of both the estimators $\tilde{\theta}_{BS}$ and $\tilde{\theta}_{BL}$ with respect D_1 and D_2 respectively under SELF higher than under LLF.

(vi) The both estimators $\tilde{\theta}_{BS}$ and $\tilde{\theta}_{BL}$ with respect $D_1$and D_2 respectively under LLF have highest relative risk when a=1.

(vii) The relative risk of estimators $\tilde{\theta}_{BS}$ with respect to $\tilde{\theta}_{BL}$ under SELF is much greater than one for all values of λ we considered except for λ=1. In the later case it is greater than one when a=0.5 and for a=1.0 it remains less than one for smaller values of m and crosses one as sample size increases.

(viii) The relative risk of estimators $\tilde{\theta}_{BL}$ with respect to $\tilde{\theta}_{BS}$ under LLF is greater than one for all values of $\lambda \ge 1$. For values of λ<1, we not able to identify any kind dominance behavior of one over the other.

As a summary to above conclusions we observe that estimators $\tilde{\theta}_{BS}$ and $\tilde{\theta}_{BL}$ perform better, in terms of relative risk, with respect to the estimators D_1 and D_2 respectively both under SELF and LLF in the neighborhood of $\theta=\theta_0$; equivalently in the neighborhood of λ=1.

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Appendix

α=0.01		λ						
m	b	0.25	0.5	0.75	1	1.25	1.5	1.75
6	2	1.0178	1.1957	1.4201	1.4922	1.3911	1.1807	0.9471
	4	0.9799	1.0914	1.6823	2.1894	1.8525	1.2431	0.8101
	8	0.9548	1.0136	1.8278	2.9341	2.1837	1.1883	0.6839
9	2	1.0058	1.1306	1.291	1.3256	1.2396	1.0619	0.8592
	4	0.9926	1.0296	1.4495	1.7772	1.5322	1.0811	0.7343
	8	0.9828	0.9551	1.5406	2.2603	1.7496	1.0266	0.6211
12	2	1.0019	1.0958	1.227	1.2431	1.1619	0.9924	0.8032
	4	0.9979	1.0073	1.3297	1.5753	1.369	0.9859	0.6869
	8	0.9947	0.9405	1.388	1.9287	1.5243	0.9313	0.5852
α=0.05								
6	2	1.0152	1.1904	1.4149	1.4244	1.2346	0.9842	0.7702
	4	1.001	1.1155	1.541	1.8641	1.494	0.9884	0.6678
	8	0.9915	1.0632	1.5989	2.2396	1.6501	0.9451	0.5867
9	2	1.0039	1.121	1.3007	1.2831	1.1084	0.8904	0.7166
	4	0.9999	1.0579	1.368	1.5875	1.2687	0.8669	0.6204
	8	0.9969	1.0117	1.3982	1.8652	1.3708	0.8211	0.546
12	2	1.001	1.0823	1.2423	1.2123	1.0442	0.8436	0.7013
	4	1	1.0335	1.2774	1.4445	1.152	0.8063	0.6105
	8	0.9992	0.9961	1.2903	1.6629	1.2218	0.7592	0.5408

Table 1: $RR_{SELF}(\tilde{\theta}_{BS}, D_1)$

a=0.5		Λ						
m	b	0.25	0.5	0.75	1	1.25	1.5	1.75
6	2	1.0877	2.4433	8.7237	16.5463	16.0006	12.5747	9.1755
	4	1.0841	2.4056	9.4502	23.0931	20.7789	13.1383	8.1087
	8	1.0815	2.3734	9.8019	29.4154	24.066	12.6438	7.0617
9	2	1.0253	1.9242	8.2498	21.0896	19.6436	13.2797	8.2133
	4	1.0246	1.9075	8.5624	26.7353	23.3324	13.4356	7.5705
	8	1.0241	1.8932	8.7213	32.1321	25.891	12.9881	6.8935
12	2	1.0071	1.616	7.5031	25.1787	21.5204	11.8374	6.2289
	4	1.007	1.6079	7.6428	30.1601	24.1344	11.8019	5.9279
	8	1.0068	1.6009	7.7148	34.893	25.9357	11.4901	5.6031
a=1.0								
6	2	1.1201	2.9441	7.2384	7.1302	6.1032	4.9158	3.7468
	4	1.1096	2.7981	8.785	10.7574	8.0528	5.1544	3.263
	8	1.1024	2.6812	9.676	14.8651	9.4305	4.9449	2.8017
9	2	1.0346	2.2237	8.1739	9.8942	8.2874	6.1433	4.2136
	4	1.0327	2.1669	9.0229	13.33	10.0275	6.2303	3.7847
	8	1.0312	2.1196	9.4975	17.0432	11.2731	5.9817	3.356
12	2	1.0097	1.7923	8.1059	12.4949	9.9222	6.3851	3.834
	4	1.0093	1.7676	8.5297	15.7423	11.3485	6.3591	3.5544
	8	1.009	1.7464	8.7594	19.1554	12.3649	6.1334	3.2685

Table 2: $RR_{SELF}(\tilde{\theta}_{BL}, D_2)$ **, α=0.01**

α=0.01		λ						
m	b	0.25	0.5	0.75	1	1.25	1.5	1.75
6	2	0.9543	1.0768	1.3984	1.4895	1.3779	1.1938	1.0024
	4	1.1025	1.74	2.3698	2.4066	1.8637	1.2512	0.8283
	8	1.2041	2.5329	3.6404	3.5632	2.2201	1.1877	0.679
9	2	0.9769	0.9869	1.2534	1.3505	1.286	1.1723	1.049
	4	1.0358	1.52	1.9537	1.9246	1.5783	1.1519	0.8258
	8	1.0769	2.2626	2.925	2.6088	1.7875	1.0614	0.6575
12	2	0.9913	0.9495	1.1723	1.2744	1.2348	1.1629	1.0861
	4	1.0115	1.3778	1.7379	1.6876	1.428	1.0922	0.8264
	8	1.026	1.9817	2.5489	2.1632	1.5605	0.9841	0.6476
α=0.05								
6	2	0.9704	1.0003	1.2581	1.4415	1.3985	1.2447	1.0767
	4	1.0554	1.5578	2.3009	2.3155	1.724	1.1675	0.8112
	8	1.1097	2.2471	4.2012	3.4299	1.8941	1.0481	0.6481
9	2	0.9888	0.9516	1.1397	1.319	1.318	1.2349	1.1357
	4	1.0159	1.361	1.9241	1.8764	1.4617	1.0668	0.8094
	8	1.0341	1.8762	3.4172	2.5479	1.5176	0.9209	0.6287
12	2	0.9966	0.9401	1.0748	1.2511	1.2726	1.2288	1.1643
	4	1.0042	1.2418	1.7206	1.656	1.3233	1.01	0.817
	8	1.0096	1.6007	2.9582	2.1269	1.3224	0.851	0.6305

Table 3: $RR_{LLF}(\tilde{\theta}_{BS}, D_1)$ **, a=1**

α=0.01		Λ						
m	b	0.25	0.5	0.75	1	1.25	1.5	1.75
6	2	0.9419	1.3041	2.872	4.4112	4.5058	4.0467	3.4801
	4	0.9832	1.527	3.9142	6.8595	6.169	4.2558	2.8277
	8	1.0072	1.6723	4.7871	9.6971	7.415	4.0243	2.285
9	2	0.9766	1.2242	3.2135	6.3072	7.0064	6.8146	6.2664
	4	0.9876	1.3379	3.9167	8.6659	8.7264	6.6803	4.7636
	8	0.9947	1.4179	4.5021	11.264	9.9888	6.0917	3.6972
12	2	0.9922	1.1576	3.3134	8.135	9.6027	9.5767	8.14
	4	0.995	1.2223	3.802	10.4075	11.2582	8.9128	6.1251
	8	0.9969	1.2703	4.2115	12.8402	12.4191	7.9211	4.7631
α=0.05								
6	2	0.9579	1.0677	1.9238	3.6121	4.7547	4.6896	4.0619
	4	0.9826	1.2144	2.4638	5.1536	5.9662	4.3468	2.9212
	8	0.997	1.3138	2.9095	6.682	6.6163	3.8326	2.2704
9	2	0.9873	1.0437	2.0038	4.8452	7.5206	7.3671	5.5238
	4	0.9925	1.1156	2.33	6.1577	8.4462	6.2239	3.9941
	8	0.9958	1.1669	2.5986	7.4095	8.8126	5.2719	3.1275
12	2	0.9967	1.0269	1.9995	5.9114	9.9221	7.9332	4.5629
	4	0.9977	1.0665	2.2157	7.0494	10.354	6.6362	3.7013
	8	0.9984	1.0959	2.396	8.1182	10.3465	5.6649	3.1177

Table 4: $RR_{LLF}(\tilde{\theta}_{BL}, D_2)$ **, a=1**

a=0.5		Λ						
m	b	0.25	0.5	0.75	1	1.25	1.5	1.75
6	2	0.0873	0.1962	0.7004	1.3284	1.2846	1.0095	0.7366
	4	0.087	0.1931	0.7587	1.854	1.6682	1.0548	0.651
	8	0.0868	0.1905	0.7869	2.3615	1.9321	1.0151	0.5669
9	2	0.0536	0.1005	0.4309	1.1016	1.0261	0.6937	0.429
	4	0.0535	0.0996	0.4473	1.3965	1.2188	0.7018	0.3954
	8	0.0535	0.0989	0.4556	1.6784	1.3524	0.6784	0.3601
12	2	0.0389	0.0625	0.2901	0.9736	0.8322	0.4577	0.2409
	4	0.0389	0.0622	0.2955	1.1663	0.9332	0.4564	0.2292
	8	0.0389	0.0619	0.2983	1.3493	1.0029	0.4443	0.2167
a=1.0								
6	2	0.2482	0.6523	1.6039	1.5799	1.3523	1.0892	0.8302
	4	0.2459	0.62	1.9466	2.3836	1.7843	1.1421	0.723
	8	0.2443	0.5941	2.144	3.2938	2.0896	1.0957	0.6208
9	2	0.1406	0.3022	1.1109	1.3446	1.1263	0.8349	0.5726
	4	0.1403	0.2945	1.2262	1.8116	1.3628	0.8467	0.5143
	8	0.1401	0.2881	1.2907	2.3162	1.532	0.8129	0.4561
12	2	0.0983	0.1744	0.7889	1.216	0.9656	0.6214	0.3731
	4	0.0982	0.172	0.8301	1.532	1.1044	0.6189	0.3459
	8	0.0982	0.17	0.8525	1.8642	1.2034	0.5969	0.3181

Table (5): $RR_{SELF}(\tilde{\theta}_{BL}, D_1)$ **, α=0.01**

α=0.01		λ						
m	b	0.25	0.5	0.75	1	1.25	1.5	1.75
6	2	1.1265	1.5874	1.448	1.3038	1.1534	0.9739	0.7993
	4	1.3809	4.5667	2.8549	2.1279	1.5448	1.0182	0.6691
	8	1.5742	99.4233	5.5787	3.1916	1.827	0.9692	0.5546
9	2	1.0265	1.4314	1.3307	1.2011	1.0898	0.9512	0.8122
	4	1.1014	3.3964	2.3461	1.7202	1.3266	0.936	0.656
	8	1.1546	20.4558	4.2926	2.3454	1.4936	0.8676	0.5328
12	2	0.9967	1.3242	1.2735	1.1503	1.0572	0.9366	0.8173
	4	1.0196	2.5695	2.1056	1.5276	1.2145	0.885	0.6465
	8	1.036	7.0421	3.6995	1.9647	1.321	0.8049	0.5207
α=0.05								
6	2	1.0522	1.5259	1.5515	1.3084	1.0959	0.9086	0.7557
	4	1.1709	4.1687	4.4197	2.1728	1.3233	0.8604	0.5966
	8	1.2498	2.9753	5.8419	3.3638	1.4384	0.7839	0.491
9	2	0.9824	1.0814	1.361	1.2981	1.1161	0.9771	0.8516
	4	0.984	1.2732	3.204	2.5572	1.4443	0.9873	0.7059
	8	0.985	1.4591	5.6707	8.7268	1.7773	0.9329	0.5741
12	2	1.0416	1.4285	2.2154	1.0092	0.8726	0.8083	0.8027
	4	1.1418	3.2425	5.654	1.0247	0.75	0.6435	0.6331
	8	1.2152	12.8937	12.9465	1.0108	0.6696	0.5502	0.5371

Table (6): $RR_{LLF}(\tilde{\theta}_{BS}, D_2)$ **, a=1**