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MOMENTS OF CONCOMITANTS OF ORDER STATISTICS FOR A NEW FINITE RANGE BI-VARIATE DISTRIBUTION (FRBD)

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Abstract

In this article mathematical expressions for moments of concomitants of order statisticshave been developed. A new bi-variate distribution has been developed for this purpose and different probability functions of concomitants have been obtained. It is purely a mathematical work that will help to study the stochastic behavior of concomitants of order statistics.

Key Words: Bi-variate distribution, order statistics, concomitant, finite range bi-variate distribution(FRBD)

1. Introduction

There are a number of univariate and as well as multivariate models in the existing literature which are widely applicable in real life situations. Bivariate with finite range are rarely seen in the literature while the applicability of such distributions cannot be ignored in real life. Especially when we deal with failure data over a finite range.

The study of concomitants has attracted many workers, Zippin and Armitage(1966) used theory of concomitants in survival analysis. Yang(1977) gave a general distribution theory on concomitants. Nagaraja and david(1994) presented the distribution of the maximum of concomitants. Nagaraja and Joshi (1995) obtained joint distribution of concomitants of order statistics. Vianna and Lee (2006) presented a study on correlation analysis usin concomitants of order statistics. Siddiqui et al(2011) developed moments and joint distribution of concomitants of order statistics.

In the present work we have tried to develop a new bivariate finite range model and the moments of concomitants have been developed this new distribution.

Development of Bivariate Models

Frechet(1951) noted that in case of bivariate distribution since

$$P[(X_1 \le x_1) \cap (X_2 \le x_2)] \le Min[P(X_1 \le x_1), P(X_2 \le X_1]$$

hence the relationship

$$F_{x_1,x_2}(X_1,X_2) \le Min[F_{x_1}(X_1), F_{x_2}(X_2) \quad \dots (1)$$

must hold for all pairs of random variables x1 and x2.
In a similar manner, since

$$P[(X_1 > x_1) \cup (X_2 > x_2)] \le [P(X_1 > x_1) + P(X_2 > X_1]$$

it follows that

$$1 - F_{x_1,x_2}(X_1,X_2) \le \left(1 - F_{x_1,}(X_1)\right) + (1 - [F_{x_2}(X_2))$$

that is

$$F_{x_1,x_2}(X_1,X_2) \ge F_{x_1,}(X_1) + F_{x_2}(X_2) - 1 \quad \dots(2)$$

Frechet(1951) suggested that any system of bivariate distributions with specified marginal distributions $F_{x_1}(X_1)$ and $F_{x_2}(X_2)$ should include the limits in (1) and (2) as limiting cases. In particular he suggested the system ;

$$F_{x_1,x_2}(X_1,X_2) = \theta \max\left[F_{x_1,}(X_1) + F_{x_2}(X_2) - 1\right] + (1-\theta)\, Min[F_{x_1}(X_1), F_{x_2}(X_2)$$

$$0 \le \theta \le 1 \quad \dots (3)$$

This system does not, however include the case when X_1 and X_2 are independent. A system that does include this case [but not the limits in (1) and (2) are given by Morgenstern (1956) as

$$F_{x_1,x_2}(X_1,X_2) = F_{x_1,}(X_1)F_{x_2}(X_2)[1 + \delta\{1 - F_{x_1,}(X_1)\}\{1 - F_{x_1,}(X_1)\}] \quad \dots (4)$$

We have used here the system given by Mongenstern(1956) to develop a bivariate finite range distribution.

2. Finite Range Bivariate Dustribution (FRBD)

Let the marginal distributions of X and Y are Mukherjee-Islam distribution [Mukherjee and Islam (1983)] with parameters (α,θ) and (β,θ). Then following the Mongenstern (1956) for δ=-1 the joint distribution function will be

$$F(x,y) = \frac{x^\alpha . y^\beta}{\theta^{\alpha+\beta}}\left[\left(\frac{x}{\theta}\right)^\alpha + \left(\frac{y}{\theta}\right)^\beta - \frac{x^\alpha . y^\beta}{\theta^{\alpha+\beta}}\right] \qquad 0<x,y<\theta \qquad \dots(5)$$

Its probability density function is obtained as

$$f(x,y) = \frac{2\alpha.\beta.x^{\alpha-1}.y^{\beta-1}}{\theta^{\alpha+\beta}}\left[\frac{x}{\theta^\alpha}^{\alpha} + \frac{y}{\theta^\beta}^{\beta} - \frac{2x^\alpha.y^\beta}{\theta^{\alpha+\beta}}\right]$$

$$0<x,y<\theta \quad \alpha,\beta > 0 \qquad \dots(6)$$

3. Marginal Probability Density Functions

The marginal probability density function of X can be obtained as;

$$g(x) = \frac{\alpha\ x^{\alpha-1}}{\theta^{\alpha}} \qquad 0 < x < \theta \qquad \text{... (7)}$$

And the marginal distribution function of X is;

$$G(x) = \left(\frac{x}{\theta}\right)^{\alpha} \qquad 0 < x < \theta \qquad \text{... (8)}$$

Similarly, the marginal probability density function of Y will be;

$$h(y) = \frac{\beta y^{\beta-1}}{\theta^{\beta}} \qquad 0 < y < \theta \qquad \text{....(9)}$$

And the marginal distribution function of Y is;

$$H(y) = \left(\frac{y}{\theta}\right)^{\theta} \qquad 0 < y < \theta \qquad \text{... (10)}$$

The mean of the random variable Y having the probability density function as defined in (5) will be;

$$E(Y) = \frac{\beta}{\beta+1}.\theta \qquad \text{... (11)}$$

Also,

$$E(Y^2) = \frac{\beta}{\beta+2}.\theta^2 \qquad \text{... (12)}$$

4. Conditional Probability Density Functions

The conditional probability density function of Y for given X can be obtained as follows;

$$h(y/x) = \frac{2\beta..y^{\beta-1}}{\theta^{\beta}}\left[\frac{x}{\theta^{\alpha}}^{\alpha} + \frac{y}{\theta^{\beta}}^{\beta} - \frac{2x^{\alpha}.y^{\beta}}{\theta^{\alpha+\beta}}\right] \qquad \text{.... (13)}$$

The conditional probability density function of X for given Y will as follows;

$$g(x/y) = \frac{2\beta y^{\beta-1}}{\theta^{\beta}}\left[\frac{x}{\theta^{\alpha}}^{\alpha} + \frac{y}{\theta^{\beta}}^{\beta} - \frac{2x^{\alpha}.y^{\beta}}{\theta^{\alpha+\beta}}\right] \qquad \text{.... (14)}$$

5. Probability Density Function of Order Statistics

The probability density function of the r^{th} order statistics $X_{r:n}$ is;

$$f_{r:n}(x) = C_{r:\mathrm{n}} \frac{\alpha\ x^{\alpha r-1}}{\theta^{\alpha r}} \left[1-\left(\frac{x}{\theta}\right)^{\alpha}\right]^{n-r} \qquad ....(15)$$

$$\text{where } C_{r:\mathrm{n}} = \frac{n!}{(r-1)!\ (n-r)!}$$

In particular for r=1, i.e. the probability density function of the first order statistics is

$$f_{1:n}(x) = n \frac{\alpha\ x^{\alpha-1}}{\theta^{\alpha}} \left[1-\left(\frac{x}{\theta}\right)^{\alpha}\right]^{n-1} \qquad ....(16)$$

For the distribution with probability density function (3), the joint distribution of two order statistics r^{th} and s^{th} is as follows;

$$f_{r,\mathrm{s}:n}(x_1, x_2) = C_{r,s:\mathrm{n}} \frac{\alpha^2\ x_1^{\alpha r-1}\ x_2^{\alpha-1}}{\theta^{\alpha(r+1)}} \left\{\left[\frac{x_2}{\theta}\right]^{\alpha} - \left[\frac{x_1}{\theta}\right]^{\alpha}\right\}^{s-r-1}$$

$$\left\{1-\left[\frac{x_2}{\theta}\right]^{\alpha}\right\}^{n-s} \qquad .... (17)$$

$$\text{where} \qquad C_{r,s:\mathrm{n}} = \frac{n!}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

6. Probability Density Function of Concomitants

The probability density function of the first order concomitant (i.e. r=1) of the order statistics is [David (1981)]

$$g_{[1:n]}(y) = \int_0^{\infty} h(y/x) f_{1:n}(x) dx$$

$$g_{[1:n]}(y) = n \sum_{k=0}^{n-1} {}^{n-1}C_k (-1)^{n-k-1} \left[\frac{2\beta y^{\beta-1}}{(n-k+1)\theta^{\beta}} + \frac{2\beta y^{2\beta-1}}{(n-k)\theta^{2\beta}} - \frac{4\beta y^{2\beta-1}}{(n-k+1)\theta^{2\beta}}\right] \qquad .... (18)$$

Similarly, $g_{[1:n]}(x)$ can be worked out as;

$$g_{[1:n]}(x) = n\sum_{k=0}^{n-1} {}^{n-1}C_k(-1)^{n-k-1}\left[\frac{2\alpha x^{\alpha-1}}{(n-k+1)\theta^{\alpha}}+\frac{2\alpha x^{2\alpha-1}}{(n-k)\theta^{2\alpha}}-\frac{4\alpha x^{2\alpha-1}}{(n-k+1)\theta^{2\alpha}}\right] \quad \ldots (19)$$

Now, the probability density function of the r[th] order concomitant can be obtained by using the following relation as;

$$g_{[r:n]}(y)= \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i} \quad g_{[1:i]}(y)$$

$$g_{[r:n]}(y)= \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}$$

$$\left\{i\sum_{k=0}^{n-1} {}^{i-1}C_k(-1)^{i-k-1}\left[\frac{2\beta y^{\beta-1}}{(i-k+1)\theta^{\beta}}+\frac{2\beta y^{2\beta-1}}{(i-k)\theta^{2\beta}}-\frac{4\beta y^{2\beta-1}}{(i-k+1)\theta^{2\beta}}\right]\right\} \quad \ldots (20)$$

Similarly, we can obtain,

$$g_{[r:n]}(x) = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i} .$$

$$\left\{i\sum_{k=0}^{n-1} {}^{i-1}C_k(-1)^{i-k-1}\left[\frac{2\alpha x^{\alpha-1}}{(i-k+1)\theta^{\alpha}}+\frac{2\alpha x^{2\alpha-1}}{(i-k)\theta^{2\alpha}}-\frac{4\alpha x^{2\alpha-1}}{(i-k+1)\theta^{2\alpha}}\right]\right\} \quad \ldots (21)$$

7. Moments of Concomitants

Once the moments are calculated then it is easy to find out any constant. The k[th] order moment will provide everything for the purpose of characterizing the concomitants.

The k[th] order moment about origin of the first concomitant i.e. of Y[1:n] is given by,

$$\mu^k_{y[1:n]} =$$

$$\int_0^{\infty} y^k \; n\sum_{j=0}^{n-1} {}^{n-1}C_j(-1)^{n-j-1}\left[\frac{2\beta y^{\beta-1}}{(n-j+1)\theta^{\beta}}+\frac{2\beta y^{2\beta-1}}{(n-j)\theta^{2\beta}}-\frac{4\beta y^{2\beta-1}}{(n-j+1)\theta^{2\beta}}\right] dy$$

Or

$\mu_{y[1:n]}^{k}$

$$= n\sum_{j=0}^{n-1} {}^{n-1}C_j (-1)^{n-j-1} \theta^k \, 2\beta \left[\frac{1}{(n-j+1)(\beta+k)} \left(\frac{1}{\beta+k} - \frac{2}{2\beta+k} \right) + \frac{1}{(n-j)(2\beta+k)} \right] \quad \ldots (22)$$

Now, the k^{th} order moment about origin of $Y_{[r:n]}$ will be;

$$\mu_{y[r:n]}^{k} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu^{k}_{[1:i]}$$

$$\mu_{y[r:n]}^{k} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i\sum_{j=0}^{i-1} {}^{i-1}C_j (-1)^{i-j-1} \theta^k \, 2\beta \left[\frac{1}{(i-j+1)(\beta+k)} \left(\frac{1}{\beta+k} - \frac{2}{2\beta+k} \right) + \frac{1}{(i-j)(2\beta+k)} \right] \right\} \quad \ldots (23)$$

Similarly, the k^{th} order moment about origin of $X_{[r:n]}$ can be obtained as ;

$$\mu_{x[r:n]}^{k} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i\sum_{j=0}^{i-1} {}^{i-1}C_j (-1)^{i-j-1} \theta^k \, 2\alpha \left[\frac{1}{(i-j+1)(\alpha+k)} \left(\frac{1}{\alpha+k} - \frac{2}{2\alpha+k} \right) + \frac{1}{(i-j)(2\alpha+k)} \right] \right\} \quad \ldots (24)$$

8. Mean and variance of Concomitants

Now, in particular for k=1 the mean of $Y_{[r:n]}$ will be;

$$\text{Mean} = E(Y_{[r:n]}) = \mu_{y[r:n]}^{1} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i\sum_{j=0}^{i-1} {}^{i-1}C_j (-1)^{i-j-1} \theta \, 2\beta \left[\frac{1}{(i-j+1)(\beta+1)} \left(\frac{1}{\beta+1} - \frac{2}{2\beta+1} \right) + \frac{1}{(i-j)(2\beta+1)} \right] \right\} \quad \ldots (25)$$

The variance of $Y_{[r:n]}$ can be obtained through the relation,

$$V(Y_{[r:n]}) = \mu_{y[r:n]}^{2} - (\mu_{y[r:n]}^{1})^2$$

where

$$\mu^2_{y[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i \sum_{j=0}^{i-1} {}^{i-1}C_j (-1)^{i-j-1} \theta^2 2\beta \left[\frac{1}{(i-j+1)(\beta+2)} \left(\frac{1}{\beta+2} - \frac{1}{\beta+1} \right) + \frac{1}{2(i-j)(\beta+1)} \right] \right\}$$

… (26)

Similarly, the mean of $X_{[r:n]}$ will be;

$$\text{Mean} = E(X_{[r:n]}) = \mu^1_{y[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i \sum_{j=0}^{i-1} {}^{i-1}C_j (-1)^{i-j-1} \theta \; 2\alpha \left[\frac{1}{(i-j+1)(\alpha+1)} \left(\frac{1}{\alpha+1} - \frac{2}{2\alpha+1} \right) + \frac{1}{(i-j)(2\alpha+1)} \right] \right\}$$

… (27)

The variance of $X_{[r:n]}$ can be obtained by using the relation,

$$V(X_{[r:n]}) = \mu^2_{x[r:n]} - (\mu^1_{x[r:n]})^2$$

Where,

$$\mu^2_{x[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i \sum_{j=0}^{i-1} {}^{i-1}C_j (-1)^{i-j-1} \theta^2 2\alpha \left[\frac{1}{(i-j+1)(\alpha+2)} \left(\frac{1}{\alpha+2} - \frac{1}{\alpha+1} \right) + \frac{1}{2(i-j)(\alpha+1)} \right] \right\}$$

… (28)

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