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IMPROVED EXPONENTIAL RATIO-TYPE ESTIMATORS IN SURVEY SAMPLING

Aamir Sanaullah, **Hina Khan, **Humera Ameer Ali and †Rajesh Singh**
*National College of Business Anministration and Economics, Lahore, Pakistan
**GC University, Lahore, Pakistan
*** Department of Statistics, BHU, Varanasi(U.P.), India
†Corresponding author

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Abstract

In this paper some improved exponential ratio-type estimators have been proposed for estimating the finite populations mean using auxiliary information on two auxiliary variables in double sampling. The properties of the proposed estimators have been analyzed for independent and dependent samples case under simple random sampling without replacement (SRSWOR). An empirical study is carried out to demonstrate the performance of proposed estimators over Noor-ul-Amin and Hanif (2012), Singh et al. (2008), Singh and Vishvakarma (2007), and Classical Ratio Estimator.

Key Words: Auxiliary Variable, Two Phase Sampling, Mean Square Error (MSE), Bias, Efficiency.

1. Introduction

In survey sampling, the use of auxiliary information has always been resulted in extensive gain in performance over the estimators which do not take such information. The auxiliary information has been effectively used in double sampling to estimate a population characteristic where the auxiliary information X is already available or can be easily observed and highly correlated with study variable Y. Ratio, product, and regression estimators are good examples in this context. Watson (1937) used regression method of estimation to estimate the average area of the leaves on the plant. Cochran (1940) used ratio method of estimation for positively correlated study variable Y and auxiliary variable X. On the other hand if this correlation is negative, Robson (1957) and Murthy (1964) suggested product method of estimation. Bhal and Tuteja (1991) suggested exponential product type and exponential ratio type estimators to estimate unknown mean of the study variable Y, when variable of interest Y and auxiliary variable X is negatively or positively correlated. Keeping these facets in consideration several authors including Sukhatme (1962), Cochran (1963), Mohanty (1967), Srivastava (1971), Khare and Srivastava (1981), Hidiroglou (2001), Singh and Espejo (2003), Samiuddin and Hanif (2007), Singh and Vishwakarma (2007), Singh et al. (2007), Misra et al. (2008), Hanif et al. (2009), Singh et al. (2009), Hanif et al. (2010), Singh et al. (2010), Singh and Kumar (2011), Singh and Smarandache(2011), Noor-ul-Amin and Hanif (2012), Singh et al. (2012) and Tailor et al. (2012) have suggested improved estimators for estimating unknown population mean of study variable Y.

Let in a finite population U={$U_1,U_2,U_3,\ldots,U_N$) of size N, $\bar{Y}=\sum_{i=1}^{N} Y_i \Big/ N$, $\bar{X}=\sum_{i=1}^{N} X_i \Big/ N$ and $\bar{Z}=\sum_{i=1}^{N} Z_i \Big/ N$ are the population means of the variable y, x and z respectively. The sample at the first phase is drawn of size 'n_1' from the population and $\bar{x}_1=\sum_{i=1}^{n_1} x_i \Big/ n_1$ be the sample mean of variable 'x' for the first phase sample, where $\bar{y}_2=\sum_{i=1}^{n_2} y_i \Big/ n_2$ and $\bar{z}_2=\sum_{i=1}^{n_2} z_i \Big/ n_2$ are the means of variable 'y' and 'z' respectively for the sample obtained at second phase of size 'n_2'. We make following assumptions:

- If second phase sample is not independent from the first phase sample.

$$
\left.
\begin{array}{ll}
\mathrm{E}(e_{\bar{y}_2})=\mathrm{E}(e_{\bar{z}_2})=\mathrm{E}(e_{\bar{z}_1})=\mathrm{E}(e_{\bar{x}_1})=0, & \\
\mathrm{E}(e_{\bar{y}_2})^2=\theta_2\bar{Y}^2C^2{}_y=\mathrm{Var}(\bar{y}), & \mathrm{E}(e_{\bar{z}_2})^2=\theta_2\bar{Z}^2C_z^2, \\
\mathrm{E}(e_{\bar{z}_1})^2=\theta_1\bar{Z}^2C_z^2, & \mathrm{E}(e_{\bar{x}_1})^2=\theta_1\bar{X}^2C_x^2, \\
\mathrm{E}(e_{\bar{y}_2}e_{\bar{z}_2})=\theta_2\overline{YZ}C_yC_z\rho_{yz}, & \mathrm{E}(e_{\bar{z}_2}e_{\bar{z}_1})=\theta_1\bar{Z}^2C_z^2, \\
\mathrm{E}(e_{\bar{z}_1}e_{\bar{x}_1})=\theta_1\overline{ZX}C_zC_x\rho_{zx}, & \mathrm{E}(e_{\bar{y}_2}e_{\bar{x}_1})=\theta_2\overline{YX}C_yC_x\rho_{yx}, \\
\theta_1=\dfrac{1-f}{n_1}, \quad \theta_2=\dfrac{1-f}{n_2}, & C_y=\dfrac{S_{xy}}{\bar{Y}}, \quad \rho_{yx}=\dfrac{S_{xy}}{S_xS_y}.
\end{array}
\right| \tag{1.1}
$$

- If the second phase sample is independent of the first phase sample, then the only difference is in covariance terms, i.e.

$$E(e_{\bar{z}_2}e_{\bar{z}_1})=0,\ \mathrm{E}(e_{\bar{y}_2}\ e_{\bar{x}_1})=0,\ \mathrm{E}(e_{\bar{z}_2}\ e_{\bar{x}_1})=0. \tag{1.2}$$

2. Some available exponential-type estimators in literature

In this section some available estimators for the population mean are reproduced.

The variance of the usual unbiased estimator $\bar{y}$ under SRSWOR scheme is:

$$Var(\bar{y})=\theta_2\bar{Y}^2C^2{}_y \tag{2.1}$$

Bahl and Tuteja (1991) suggested exponential ratio-type estimator for single phase sampling as:

$$t_1=\bar{y}\ \exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{2.2}$$

The mean square error of t_1 is

$$\text{MSE}(t_1) \approx \bar{Y}^2\left[\theta C_y^2 + \frac{\theta C_x^2}{4} + \theta\rho_{xy}C_yC_x\right]. \tag{2.3}$$

The two phase version of classical ratio estimator, when the information on population mean of auxiliary variable is known is:

$$t_2 = \bar{y}_2\frac{\bar{X}}{\bar{x}_2} \tag{2.4}$$

The mean square error of the estimator t_2 is

$$MSE(t_2) \approx \theta_2\bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x\right) \tag{2.5}$$

Singh and Vishwakarma (2007) suggested exponential ratio-type estimator in double sampling as:

$$t_3 = \bar{y}_2 \exp\left[\frac{\bar{x}_1 - \bar{x}_2}{\bar{x}_1 + \bar{x}_2}\right] \tag{2.6}$$

The MSE of the estimator t_3 is

$$\text{MSE}(t_3) \approx S_y^2\left(\theta_2 + \frac{C_x}{4C_y}(\theta_2 - \theta_1)\left(\frac{C_x}{C_y} - \rho_{xy}\right)\right) \tag{2.7}$$

Estimator t_3 is the modified form of t_1. Singh and Vishwakarma in (2007) has proved that the performance of t_3 is better than t_1.

Singh and Vishwakarma (2007) have also suggested exponential product-type estimator for double sampling as:

$$t_4 = \bar{y}_2 \exp\left[\frac{\bar{x}_2 - \bar{x}_1}{\bar{x}_2 + \bar{x}_1}\right] \tag{2.8}$$

The MSE of the estimator t_4 is given by:

$$MSE(t_4) \approx S_y^2\left(\theta_2 + \frac{C_x}{4C_y}(\theta_2 - \theta_1)\left(\frac{C_x}{C_y} + 4\rho_{xy}\right)\right) \tag{2.9}$$

Singh et al. (2008) suggested an exponential ratio-type estimator in double sampling as:

$$t_5 = w_{od}\bar{y}_2 + w_{1d}\bar{y}_{rsd} + w_{2d}\bar{y}_{rsed} \tag{2.10}$$

where

$$\sum_{i=0}^{2} w_{id} = 1, t_{rsd} = \bar{y}_2 \exp\left[\frac{a\bar{x}_1 + b}{a\bar{x}_2 + b}\right], \qquad t_{rsd} = \bar{y}_2 \exp\left[\frac{(a\bar{x}_1 + b) - (a\bar{x}_2 + b)}{(a\bar{x}_1 + b) + (a\bar{x}_2 + b)}\right].$$

The minimum MSE of the estimator t_5 is given by:

$$MSE(t_5) \approx \bar{Y}^2C_y^2\left(\theta_2 - (\theta_2 - \theta_1)\rho_{xy}^2\right) \tag{2.11}$$

Singh et al. (2008) also suggested an exponential product-type estimator in double sampling as:

$$t_6 = q_{od}\bar{y}_2 + q_{1d}\bar{y}_{rsd} + q_{2d}\bar{y}_{rsed} \tag{2.12}$$

where

$$\sum_{i=0}^{2} q_{id} = 1, t_{rsd} = \bar{y}_2 \exp\left[\frac{a\bar{x}_2 + b}{a\bar{x}_1 + b}\right], \qquad t_{rsd} = \bar{y}_2 \exp\left[\frac{(a\bar{x}_2 + b) - (a\bar{x}_1 + b)}{(a\bar{x}_1 + b) + (a\bar{x}_2 + b)}\right].$$

The minimum MSE of the estimator t_6 is given by:

$$MSE(t_6) \approx \bar{Y}^2 C_y^2 \left(\theta_2 - (\theta_2 - \theta_1)\rho_{xy}^2\right). \tag{2.13}$$

Note that the minimum MSE of the estimator t_6 is equivalent to the minimum MSE of the estimator t_5.

Noor-ul-Amin and Hanif (2012) suggested ratio-cum-product type exponential estimator for double sampling as:

$$t_7 = \bar{y}_2 \exp\left[\frac{\bar{Z} - \bar{z}_2}{\bar{Z} + \bar{z}_2} - \frac{\bar{X} - \bar{x}_1}{\bar{X} + \bar{x}_1}\right]. \tag{2.14}$$

Case-I (when 2nd phase sample is dependent of 1st phase sample)

The minimum MSE of the estimator t_7 is given by:

$$MSE(t_7) \approx \bar{Y}^2\left[\frac{\theta_2}{4}\left(4C_y^2 + C_z^2 - 4\rho_{yz}C_yC_z\right) + \frac{\theta_1}{4}\left(C_x^2 + 4\rho_{yx}C_yC_x - 2\rho_{zx}C_zC_x\right)\right] \tag{2.15}$$

Case-II (when 2nd phase sample is independent of 1st phase sample)

In this case the minimum MSE of the estimator t_7 is given by:

$$MSE(t_7) \approx \frac{\bar{Y}^2}{4}\left[\theta_1 C_x^2 + \theta_2\left(4C_y^2 + C_z^2 - 4\rho_{yz}C_yC_z\right)\right] \tag{2.16}$$

Noor-ul-Amin and Hanif (2012) suggested chain ratio type exponential estimator for double sampling as:

$$t_8 = \bar{y}_2 \exp\left[\frac{\bar{X} - \bar{x}_1}{\bar{X} + \bar{x}_1} + \frac{\bar{Z} - \bar{z}_2}{\bar{Z} + \bar{z}_2}\right] \tag{2.17}$$

Case-I (when 2nd phase sample is dependent of 1st phase sample)

The minimum MSE of the estimator t_8 is given by:

$$MSE(t_8) \approx \bar{Y}^2\left[\frac{\theta_2}{4}\left(4C_y^2 + C_z^2 - 4\rho_{yz}C_yC_z\right) + \frac{\theta_1}{4}\left(C_x^2 - 4\rho_{yx}C_yC_x + 2\rho_{zx}C_zC_x\right)\right] \tag{2.18}$$

Case-II (when 2nd phase sample is independent of 1st phase sample)

$$MSE(t_8) \approx \frac{\bar{Y}^2}{4}\left[\theta_1 C_x^2 + \theta_2\left(4C_y^2 + C_z^2 - 4\rho_{yz}C_yC_z\right)\right] \tag{2.19}$$

3. Proposed estimators

In this section some new exponential estimators using two auxiliary variables have been proposed. The MSE for the suggested estimators has been derived under the following two situations:

(i) When the second phase sample of size 'n_2' is dependent on the first phase sample of size 'n_1', i.e. the second phase sample is a sub-sample of the first phase sample.

(ii) When the second phase sample of size 'n_2' is independent from the first phase sample of size 'n_1'.

3.1 Exponential Estimator-I

We propose estimator-I denoted by t_9 as:

$$t_9 = \bar{y}_2 \exp\left[\alpha \frac{\bar{Z} - \bar{z}_2}{\bar{Z} + \bar{z}_2} - (1-\alpha)\frac{\bar{X} - \bar{x}_1}{\bar{X} + \bar{x}_1}\right] \tag{3.1}$$

3.11 Case-I (when sample at 2nd phase is dependent of the sample at 1st phase)

Expressing t_9 in terms of e's, we have

$$t_9 \approx \bar{Y} + e_{\bar{y}_2} - \alpha \frac{\bar{Y}}{\bar{Z}} e_{\bar{z}_2} + (1-\alpha)\frac{\bar{Y}}{2\bar{X}} e_{\bar{x}_1} \tag{3.2}$$

Expanding the right hand side and neglecting the terms of e's with power two or greater, we get

$$t_9 - \bar{Y} \approx e_{\bar{y}_2} - \alpha \frac{\bar{Y}}{\bar{Z}} e_{\bar{z}_2} + (1-\alpha)\frac{\bar{Y}}{2\bar{X}} e_{\bar{x}_1} \tag{3.3}$$

Squaring both sides and taking the expectations, we get

$$\text{MSE}(t_9) \approx E(t_9 - \bar{Y})^2 = E\left(e_{\bar{y}_2} - \alpha \frac{\bar{Y}}{\bar{Z}} e_{\bar{z}_2} + (1-\alpha)\frac{\bar{Y}}{2\bar{X}} e_{\bar{x}_1} \right)^2 \tag{3.4}$$

On simplification we get the MSE of t_9 as:

$$MSE(t_9) \approx \theta_2 \bar{Y}^2 \left[C_y^2 + \alpha^2 C_z^2 - 2\alpha C_y C_z \rho_{yz}\right] + \theta_1 \bar{Y}^2 (1-\alpha)\left[(\frac{1-\alpha}{4})C_x^2 + C_x C_y \rho_{xy} - \alpha C_x C_z \rho_{xz}\right] \tag{3.5}$$

To obtain the optimum value of α, we partially differentiate the expression (3.5) with respect to α and put it equal to zero, we get

$$\hat{\alpha} = \frac{2\theta_2 m_1 + \theta_1(m_2 - m_3 - m_4)}{2\theta_2 m_5 + \theta_1(m_2 - 2m_4)}, \; where \begin{cases} m_1 = C_y C_z \rho_{yz}, \; m_2 = \dfrac{C_x^2}{2}, \; m_3 = C_x C_y \rho_{xy} \\ m_4 = C_x C_z \rho_{xz} \; m_5 = C_z^2 \end{cases}$$

Now, putting the optimum value of α in expression (3.5) we get the min.MSE(t_9) as:

$$\text{MSE}(t_9)\big|_{\alpha=\hat{\alpha}} = \text{min.MESE}(t_9) \approx \theta_2 \bar{Y}^2 \left[C_y^2 + \hat{\alpha}^2 C_z^2 - 2\hat{\alpha} C_y C_z \rho_{yz}\right]^2$$
$$+ \theta_1 \bar{Y}^2 (1-\hat{\alpha})\left[(\frac{1-\hat{\alpha}}{4})C_x^2 + C_x C_y \rho_{xy} - \hat{\alpha} C_x C_z \rho_{xz}\right]$$

order to derive the bias of t_9, we use expression (3.2) upto 2nd order approximation

$$t_9 \approx \bar{Y} + \bar{Y}\left[\alpha \frac{e^2_{\bar{z}_2}}{4\bar{Z}^2} - (1-\alpha)\frac{e_{\bar{x}_1} e_{\bar{x}_2}}{4\bar{X}^2}\right] - \alpha \frac{e_{\bar{y}_2} e_{\bar{z}_2}}{2\bar{Z}} + (1-\alpha)\frac{e_{\bar{x}_1} e_{\bar{y}_2}}{2\bar{X}}. \quad (3.6)$$

Or

$$t_9 - \bar{Y} \approx \bar{Y}\left[\alpha \frac{e^2_{\bar{z}_2}}{4\bar{Z}^2} - (1-\alpha)\frac{e_{\bar{x}_1} e_{\bar{x}_2}}{4\bar{X}^2}\right] - \alpha \frac{e_{\bar{y}_2} e_{\bar{z}_2}}{2\bar{Z}} + (1-\alpha)\frac{e_{\bar{x}_1} e_{\bar{y}_2}}{2\bar{X}} \quad (3.7)$$

Taking expectations and on simplification, the bias of t_9 is given by:

$$Bias(t_9) \approx \bar{Y}\left[\theta_2 \hat{\alpha}\left(\frac{C_z^2}{4} - \frac{C_y C_z \rho_{yz}}{2}\right) - \theta_1 (1-\hat{\alpha})\left(\frac{C_x^2}{4} - \frac{C_x C_y \rho_{xy}}{2}\right)\right] \quad (3.9)$$

3.1.1 Case-II (when sample at 2nd phase is independent of the sample at 1st phase)

Using the notations given in (1.1) and (1.2), t_9 may be written as

$$t_9 \approx \bar{Y} + e_{\bar{y}_2} - \alpha \frac{\bar{Y}}{\bar{Z}} e_{\bar{z}_2} + (1-\alpha)\frac{\bar{Y}}{2\bar{X}} e_{\bar{x}_1} \quad (3.10)$$

Expanding the right hand side and neglecting the terms with power two or greater, we get

$$t_9 - \bar{Y} \approx e_{\bar{y}_2} - \alpha \frac{\bar{Y}}{\bar{Z}} e_{\bar{z}_2} + (1-\alpha)\frac{\bar{Y}}{2\bar{X}} e_{\bar{x}_1}. \quad (3.11)$$

Squaring both sides and taking the expectations, we get

$$\text{MSE}(t_9) \approx E(t_9 - \bar{Y})^2 = E\left(e_{\bar{y}_2} - \alpha \frac{\bar{Y}}{\bar{Z}} e_{\bar{z}_2} + (1-\alpha)\frac{\bar{Y}}{2\bar{X}} e_{\bar{x}_1}\right)^2 \quad (3.12)$$

Expanding the right hand side, taking expectations and on simplification, we get the MSE of t_9 as:

$$MSE(t_9) \approx \theta_2 \bar{Y}^2\left[C_y^2 + \alpha^2 C_z^2 - 2\alpha C_y C_z \rho_{yz}\right] + \theta_1 \bar{Y}^2 \left(\frac{1-\alpha}{2}\right)^2 C_x^2 \quad (3.13)$$

To obtain the optimum value of α, we partially differentiate the expression (3.13) with respect to α and put it equal to zero, we get

$$\hat{\alpha} = \frac{\theta_2 k_1 + \theta_1 k_2}{\theta_2 k_3 + \theta_1 k_2}, \quad where \{ k_1 = 4C_y C_z \rho_{yz}, \quad k_2 = C_x^2, \quad k_3 = 4C_z^2$$

Now, we get min.MSE(t_9) as:

$$MSE(t_9)\big|_{\alpha=\hat{\alpha}} = \min.MSE(t_9) \approx \theta_2 \bar{Y}^2\left[C_y^2 + \hat{\alpha}^2 C_z^2 - 2\hat{\alpha} C_y C_z \rho_{yz}\right] + \theta_1 \bar{Y}^2 \left(\frac{1-\hat{\alpha}}{2}\right)^2 C_x^2 \quad (3.14)$$

In order to derive the bias of t_9, we use expression (3.2) upto 2nd order of approximation and we get

$$t_9 \approx \bar{Y} + \bar{Y}\left[\alpha\frac{e^2_{\bar{z}_2}}{4\bar{Z}^2} - (1-\alpha)\frac{e_{\bar{x}_1}e_{\bar{x}_2}}{4\bar{X}^2}\right] - \alpha\frac{e_{\bar{y}_2}e_{\bar{z}_2}}{2\bar{Z}} + (1-\alpha)\frac{e_{\bar{x}_1}e_{\bar{y}_2}}{2\bar{X}} \quad (3.15)$$

Or

$$t_9 - \bar{Y} \approx \bar{Y}\left[\alpha\frac{e^2_{\bar{z}_2}}{4\bar{Z}^2} - (1-\alpha)\frac{e_{\bar{x}_1}e_{\bar{x}_2}}{4\bar{X}^2}\right] - \alpha\frac{e_{\bar{y}_2}e_{\bar{z}_2}}{2\bar{Z}} + (1-\alpha)\frac{e_{\bar{x}_1}e_{\bar{y}_2}}{2\bar{X}} \quad (3.16)$$

Taking expectations and on simplification, the bias of t_9 is given by

$$Bias(t_9) \approx \bar{Y}\theta_2\hat{\alpha}\left(\frac{C_z^2}{4} - \frac{C_y C_z \rho_{yz}}{2}\right) \quad (3.17)$$

3.2 Family of Deduced Class of Estimators of t_9

A number of estimators can be deduced as a family of deduced estimators of t9 on setting different values of α.

Remark 3.2.1

On setting the values of $\alpha = \sqrt{\rho_{xy}}$ in (3.1) we deduce an estimator t_{10} as following

$$t_{10} = \bar{y}_2 \exp\left[\frac{\sqrt{\rho_{xy}}(\bar{Z} - \bar{z}_2)}{\bar{Z} + \bar{z}_2} - \frac{(1-\sqrt{\rho_{xy}})(\bar{X} - \bar{x}_1)}{\bar{X} + \bar{x}_1}\right] \quad \forall\, \rho_{ij} > o,\ where\ i \neq j \quad (3.18)$$

3.2.1.1 Case-I (when sample at 2nd phase is dependent of the sample at 1st phase)

$$\mathrm{MSE}(t_{10}) \approx \theta_2\bar{Y}^2\left[C_y^2 + \rho_{xy}\, C_z^2 - 2\sqrt{\rho_{xy}}C_yC_z\rho_{yz}\right] + \theta_1\bar{Y}^2(1-\sqrt{\rho_{xy}})\left[(\frac{1-\sqrt{\rho_{xy}}}{4})C_x^2 + C_xC_y\rho_{xy} - \sqrt{\rho_{xy}}C_xC_z\rho_{xz}\right] \quad (3.19)$$

$$Bias(t_{10}) \approx \bar{Y}\left[\theta_2\sqrt{\rho_{xy}}\left(\frac{C_z^2}{4} - \frac{C_yC_z\rho_{yz}}{2}\right) - \theta_1(1-\sqrt{\rho_{xy}})\left(\frac{C_x^2}{4} - \frac{C_xC_y\rho_{xy}}{2}\right)\right] \quad (3.20)$$

3.2.1.2 Case-II (when sample at 2nd phase is independent of the sample at 1st phase)

$$MSE(t_{10}) \approx \theta_2\bar{Y}^2\left[C_y^2 + \rho_{xy}\, C_z^2 - 2\sqrt{\rho_{xy}}C_yC_z\rho_{yz}\right] + \theta_1\bar{Y}^2\left(\frac{1-\sqrt{\rho_{xy}}}{2}\right)^2 C_x^2 \quad (3.21)$$

$$Bias(t_{10}) \approx \bar{Y}\theta_2\sqrt{\rho_{xy}}\left(\frac{C_z^2}{4}-\frac{C_yC_z\rho_{yz}}{2}\right)$$

(3.22)

Remark 3.2.2

On setting the values of $\alpha = \sqrt{\rho_{yz}}$ in (3.1) we get another estimator t_{11} as follows

$$t_{11}=\bar{y}_2\exp\left[\frac{\sqrt{\rho_{yz}}(\bar{Z}-\bar{z}_2)}{\bar{Z}+\bar{z}_2}-\frac{(1-\sqrt{\rho_{yz}})(\bar{X}-\bar{x}_1)}{\bar{X}+\bar{x}_1}\right] \quad \forall\rho_{ij}>o,\ where\ i\neq j \tag{3.23}$$

3.2.2.1 Case-I (when sample at 2nd phase is dependent of the sample at 1st phase)

$$MSE(t_{11})=\theta_2\bar{Y}^2\left[C_y^2+\rho_{yz}C_z^2-2C_yC_z\rho_{yz}^{3/2}\right]+\theta_1\bar{Y}^2(1-\sqrt{\rho_{yz}})$$
$$\left[\left(\frac{1-\sqrt{\rho_{yz}}}{4}\right)C_x^2+C_xC_y\rho_{xy}-\sqrt{\rho_{yz}}C_xC_z\rho_{xz}\right] \tag{3.24}$$

$$Bias(t_{11})\approx\bar{Y}\left[\theta_2\sqrt{\rho_{yz}}\left(\frac{C_z^2}{4}-\frac{C_yC_z\rho_{yz}}{2}\right)-\theta_1(1-\sqrt{\rho_{yz}})\left(\frac{C_x^2}{4}-\frac{C_xC_y\rho_{xy}}{2}\right)\right] \tag{3.25}$$

3.2.2.2 Case-II (when sample at 2nd phase is independent of the sample at 1st phase)

$$MSE(t_{11})\approx\theta_2\bar{Y}^2\left[C_y^2+\rho_{yz}C_z^2-2\rho_{yz}^{3/2}C_yC_z\right]+\theta_1\bar{Y}^2\left(\frac{1-\sqrt{\rho_{yz}}}{2}\right)^2C_x^2 \tag{3.26}$$

$$Bias(t_{11})\approx\bar{Y}\theta_2\sqrt{\rho_{yz}}\left(\frac{C_z^2}{4}-\frac{C_yC_z\rho_{yz}}{2}\right) \tag{3.27}$$

Remark 3.2.3

On setting the values of $\alpha = \sqrt{\rho_{zx}}$ in (3.1) we deduce an estimator t_{12} as follows

$$t_{12}=\bar{y}_2\exp\left[\frac{\sqrt{\rho_{zx}}(\bar{Z}-\bar{z}_2)}{\bar{Z}+\bar{z}_2}-\frac{(1-\sqrt{\rho_{zx}})(\bar{X}-\bar{x}_1)}{\bar{X}+\bar{x}_1}\right] \quad \forall\rho_{ij}>o,\ where\ i\neq j \tag{3.28}$$

3.2.3.1 Case-I (when sample at 2[nd] phase is dependent of the sample at 1[st] phase)

$$MSE(t_{12}) \approx \theta_2 \overline{Y}^2 \left[C_y^2 + \rho_{zx} C_z^2 - 2\sqrt{\rho_{zx}} C_y C_z \rho_{yz} \right] + \theta_1 \overline{Y}^2 (1 - \sqrt{\rho_{zx}}) \left[\left(\frac{1-\sqrt{\rho_{zx}}}{4} \right) C_x^2 + C_x C_y \rho_{xy} - \sqrt{\rho_{zx}} C_x C_z \rho_{xz} \right] \tag{3.29}$$

$$Bias(t_{12}) \approx \overline{Y} \left[\theta_2 \sqrt{\rho_{zx}} \left(\frac{C_z^2}{4} - \frac{C_y C_z \rho_{yz}}{2} \right) - \theta_1 (1 - \sqrt{\rho_{zx}}) \left(\frac{C_x^2}{4} - \frac{C_x C_y \rho_{xy}}{2} \right) \right] \tag{3.30}$$

3.2.3.2 Case-II (when sample at 2[nd] phase is independent of the sample at 1[st] phase)

$$MSE(t_{12}) \approx \theta_2 \overline{Y}^2 \left[C_y^2 + \rho_{zx} C_z^2 - 2\sqrt{\rho_{zx}} C_y C_z \rho_{yz} \right] + \theta_1 \overline{Y}^2 \left(\frac{1-\sqrt{\rho_{zx}}}{2} \right)^2 C_x^2 \tag{3.31}$$

$$Bias(t_{12}) \approx \overline{Y} \theta_2 \sqrt{\rho_{zx}} \left(\frac{C_z^2}{4} - \frac{C_y C_z \rho_{yz}}{2} \right) \tag{3.32}$$

Two more estimators are deduced as follows:

Deduced Estimator	$\alpha = \sqrt{\rho_{ij}}$	Sampling	Mean Square and Bias
$t_{13} = \overline{y}_2 \exp\left[\frac{\overline{Z} - \overline{z}_2}{\overline{Z} + \overline{z}_2} \right]$ Bhal and Tuteja (1991) type ratio estimator	1	Dependent	$\min .MESE(t_{13}) \approx \theta_2 \overline{Y}^2 \left[C_y^2 + C_z^2 - 2C_y C_z \rho_{yz} \right]$ $Bias(t_{13}) \approx \overline{Y}\theta_2 \left(\frac{C_z^2}{4} - \frac{C_y C_z \rho_{yz}}{2} \right)$
		Independent	$\min .MSE(t_{13}) \approx \theta_2 \overline{Y}^2 \left[C_y^2 + C_z^2 - 2C_y C_z \rho_{yz} \right]$ $Bias(t_{13}) \approx \overline{Y}\theta_2 \left(\frac{C_z^2}{4} - \frac{C_y C_z \rho_{yz}}{2} \right)$
$t_{14} = \overline{y}_2 \exp\left[\frac{\overline{x}_1 - \overline{X}}{\overline{x}_1 + \overline{X}} \right]$ Bhal and Tuteja (1991) type product estimator	0	Dependent	$\min .MESE(t_{14}) \approx \theta_2 \overline{Y}^2 C_y^2 + \theta_1 \overline{Y}^2 \left[\frac{C_x^2}{4} + C_x C_y \rho_{xy} \right]$ $Bias(t_{14}) \approx \overline{Y}\theta_1 \left(\frac{C_x^2}{4} - \frac{C_x C_y \rho_{xy}}{2} \right)$
		Independent	$\min .MSE(t_{14}) \approx \theta_2 \overline{Y}^2 C_y^2 + \frac{\theta_1 \overline{Y}^2 C_x^2}{4}$ $Bias(t_{14}) \approx 0$

3.5 Exponential Estimator-V

We propose another improved estimator-V as:

$$t_{15} = \bar{y}_2\left[\frac{\bar{X}-\bar{x}_1}{\bar{X}+\bar{x}_1} + \exp\left(\frac{(\bar{Z}-\bar{z}_2)}{\bar{Z}+\bar{z}_2} + \frac{(\bar{x}_1-\bar{x}_2)}{\bar{x}_1+\bar{x}_2}\right)\right] \quad (3.33)$$

3.5.1 Case-I (when sample at 2nd phase is dependent of the sample at 1st phase)

Expressing (3.33) in terms of e's, we have

$$t_{15} \approx \bar{Y} + e_{\bar{y}_2} - \frac{\bar{Y}}{2\bar{Z}}e_{\bar{z}_2} - \frac{\bar{Y}}{2\bar{X}}e_{\bar{x}_2} \quad (3.34)$$

Expanding the right hand side and neglecting the terms of e's with power two or greater, we get

$$t_{15} - \bar{Y} \approx e_{\bar{y}_2} - \frac{\bar{Y}}{2\bar{Z}}e_{\bar{z}_2} - \frac{\bar{Y}}{2\bar{X}}e_{\bar{x}_2} \quad (3.35)$$

Squaring both sides and taking the expectations, we get

$$\text{MSE}\,(t_{15}) \approx E(t_{15}-\bar{Y})^2 = E\left(e_{\bar{y}_2} - \frac{\bar{Y}}{2\bar{Z}}e_{\bar{z}_2} - \frac{\bar{Y}}{2\bar{X}}e_{\bar{x}_2}\right)^2 \quad (3.36)$$

Expanding the right hand side, taking expectations and on simplification, we get MSE of t_{15} as:

$$MSE(t_{15}) = \theta_2\bar{Y}^2\left[C_y^2 + \frac{C_z^2}{4} + \frac{C_x^2}{4} - C_yC_z\rho_{yz} - C_xC_y\rho_{xy} + \frac{C_xC_z\rho_{xz}}{2}\right] \quad (3.37)$$

In order to derive the bias of t_{15}, we again (3.35) upto 2nd order approximation and we get

$$t_{15} \approx \bar{Y} + e_{\bar{y}_2} + \bar{Y}\left[\begin{array}{l}\frac{3e^2_{\bar{z}_2}}{8\bar{Z}^2} - \frac{e_{\bar{z}_2}}{2\bar{Z}} - \frac{e^2_{\bar{x}_2}}{8\bar{X}^2} - \frac{e_{\bar{x}_2}}{2\bar{X}} + \frac{e_{\bar{x}_1}e_{\bar{x}_2}}{4\bar{X}^2} + \frac{e^2_{\bar{x}_1}}{8\bar{X}^2} \\ \qquad - \frac{e_{\bar{x}_1}e_{\bar{z}_2}}{4\bar{Z}\bar{X}} + \frac{e_{\bar{x}_2}e_{\bar{z}_2}}{4\bar{Z}\bar{X}}\end{array}\right] - \frac{e_{\bar{y}_2}e_{\bar{z}_2}}{2\bar{Z}} \quad (3.38)$$

Or

$$t_{15} - \bar{Y} \approx e_{\bar{y}_2} + \bar{Y}\left[\begin{array}{l}\frac{3e^2_{\bar{z}_2}}{8\bar{Z}^2} - \frac{e_{\bar{z}_2}}{2\bar{Z}} - \frac{e^2_{\bar{x}_2}}{8\bar{X}^2} - \frac{e_{\bar{x}_2}}{2\bar{X}} + \frac{e_{\bar{x}_1}e_{\bar{x}_2}}{4\bar{X}^2} + \frac{e^2_{\bar{x}_1}}{8\bar{X}^2} \\ - \frac{e_{\bar{x}_1}e_{\bar{z}_2}}{4\bar{Z}\bar{X}} + \frac{e_{\bar{x}_2}e_{\bar{z}_2}}{4\bar{Z}\bar{X}}\end{array}\right] - \frac{e_{\bar{y}_2}e_{\bar{z}_2}}{2\bar{Z}} \quad (3.39)$$

Taking expectations, and on simplification, the bias of t_{15} is given by

$$Bias(t_{15}) \approx \overline{Y}\left[\begin{array}{l}\theta_2\left(\frac{3C_z^2}{8}-\frac{C_x^2}{8}+\frac{C_xC_z\rho_{xz}}{4}-\frac{C_yC_z\rho_{yz}}{2}\right)\\ +\theta_1\left(\frac{3C_x^2}{8}-\frac{C_xC_z\rho_{xz}}{4}\right)\end{array}\right] \tag{3.40}$$

3.5.2 Case-II (when sample at 2nd phase is independent of the sample at 1st phase)

Using the notations given in (1.1) and (1.2), t_{15} may be written as

$$t_{15} \approx \overline{Y}+e_{\overline{y}_2}-\frac{\overline{Y}}{2\overline{Z}}e_{\overline{z}_2}-\frac{\overline{Y}}{2\overline{X}}e_{\overline{x}_2} \tag{3.41}$$

Expanding the right hand side and neglecting the terms of e's with power two or greater, we get

$$t_{15}-\overline{Y} \approx e_{\overline{y}_2}-\frac{\overline{Y}}{2\overline{Z}}e_{\overline{z}_2}-\frac{\overline{Y}}{2\overline{X}}e_{\overline{x}_2} \tag{3.42}$$

Squaring both sides and taking the expectations, we get

$$\text{MSE}(t_{15}) \approx E(t_{15}-\overline{Y})^2 = E\left(e_{\overline{y}_2}-\frac{\overline{Y}}{2\overline{Z}}e_{\overline{z}_2}-\frac{\overline{Y}}{2\overline{X}}e_{\overline{x}_2}\right)^2 \tag{3.43}$$

Expanding the right hand side, applying expectation and on simplification we get MSE of t_{15}

$$MSE(t_{15}) = \theta_2\overline{Y}^2\left[C_y^2+\frac{C_z^2}{4}+\frac{C_x^2}{4}-C_yC_z\rho_{yz}-C_xC_y\rho_{xy}+\frac{C_xC_z\rho_{xz}}{2}\right] \tag{3.44}$$

MSE(t_{15}) is similar for both samples (dependent and independent).

In order to derive the bias of t_{15}, from (3. 41), we get

$$t_{15} \approx \overline{Y}+e_{\overline{y}_2}+\overline{Y}\left[\begin{array}{r}\frac{3e^2_{\overline{z}_2}}{8\overline{Z}^2}-\frac{e_{\overline{z}_2}}{2\overline{Z}}-\frac{e^2_{\overline{x}_2}}{8\overline{X}^2}-\frac{e_{\overline{x}_2}}{2\overline{X}}+\frac{e_{\overline{x}_1}e_{\overline{x}_2}}{4\overline{X}^2}+\frac{e^2_{\overline{x}_1}}{8\overline{X}^2}\\ -\frac{e_{\overline{x}_1}e_{\overline{z}_2}}{4\overline{Z}\overline{X}}+\frac{e_{\overline{x}_2}e_{\overline{z}_2}}{4\overline{Z}\overline{X}}\end{array}\right]-\frac{e_{\overline{y}_2}e_{\overline{z}_2}}{2\overline{Z}} \tag{3.45}$$

Or

$$t_{15}-\overline{Y} \approx e_{\overline{y}_2}+\overline{Y}\left[\begin{array}{l}\frac{3e^2_{\overline{z}_2}}{8\overline{Z}^2}-\frac{e_{\overline{z}_2}}{2\overline{Z}}-\frac{e^2_{\overline{x}_2}}{8\overline{X}^2}-\frac{e_{\overline{x}_2}}{2\overline{X}}+\frac{e_{\overline{x}_1}e_{\overline{x}_2}}{4\overline{X}^2}+\\ \frac{e^2_{\overline{x}_1}}{8\overline{X}^2}-\frac{e_{\overline{x}_1}e_{\overline{z}_2}}{4\overline{Z}\overline{X}}+\frac{e_{\overline{x}_2}e_{\overline{z}_2}}{4\overline{Z}\overline{X}}\end{array}\right]-\frac{e_{\overline{y}_2}e_{\overline{z}_2}}{2\overline{Z}} \tag{3.46}$$

Taking expectations and on simplification, the bias of t_{15} is written as:

$$Bias(t_{15}) \approx \bar{Y}\left[\theta_2\left(\frac{3C_z^2}{8} - \frac{C_x^2}{8} + \frac{C_x C_z \rho_{xz}}{4} - \frac{C_y C_z \rho_{yz}}{2}\right) + \frac{\theta_1 C_x^2}{8}\right] \quad (3.47)$$

4. Empirical study

In order to examine the performance of proposed estimators, we have taken the populations of three different districts of Punjab, from the population census report (1998). The comparison of proposed estimators with respect to usual unbiased estimator $\bar{y}$, classical ratio estimator, Noor-ul-Amin and Hanif (2012), Singh et al. (2008), and Singh and Vishvakarma (2007) estimators have been made. The description of populations is given below:

Population I (Lodhran):

θ_1 = 0.006, θ_2 = 0.041, $\bar{Y}$ = 300.517, $\bar{X}$ = 2373.697, $\bar{Z}$ =149.09, C_x =1.076, C_y =1.097, C_z =1.331, ρ_{xy} =0.814, ρ_{yz} =0.891, ρ_{xz} =0.861

Population II (Multan):

θ_1 = 0.009, θ_2 = 0.033, $\bar{Y}$ = 646.215, $\bar{X}$ = 4533.981, $\bar{Z}$ =325.035, C_x =1.342, C_y =1.509, C_z =1.335, ρ_{xy} =0.623, ρ_{yz} =0.907, ρ_{xz} =0.682

Population III (Muzaffargarh):

θ_1 = 0.004, θ_2 = 0.028, $\bar{Y}$ = 279, $\bar{X}$ = 2411, $\bar{Z}$ =145, C_x =0.968, C_y =1.5, C_z =1.62, ρ_{xy} =0.641, ρ_{yz} =0.839, ρ_{xz} =0.673

	Case-I (Dependent Sample)				**Case-II (Independent Sample)**		
	Population #				**Population #**		
Estimator	**1**	**2**	**3**		**1**	**2**	**3**
Proposed Estimator : t_9	473.41	563.83	334.84		306.13	563.75	527.73
Proposed Estimator: t_{10}	406.28	433.27	335.10		305.60	447.82	443.89
Proposed Estimator: t_{11}	371.06	548.89	313.91		285.05	550.58	406.28
Proposed Estimator: t_{12}	384.85	464.32	333.88		304.13	475.96	420.95
Proposed Estimator: t_{15}	414.24	327.63	322.10		292.67	327.63	455.90

Classical Ratio Estimator t_1	273.79	146.46	169.74		273.79	146.46	169.74
Singh & Vishwakarma(2007) t_3	31.45	134.98	136.11		128.82	143.35	230.58
Singh et al. (2008) t_5	35.43	139.33	154.36		*	*	*
Noor-ul-Amin and Hanif (2012) t_7	31.69	190.43	234.77		226.96	223.62	341.61
Noor-ul-Amin and Hanif (2012) t_8	47.33	270.82	266.84		226.96	223.62	341.61

* Data not applicable

Table 4.1: Percent relative efficiencies of estimators with respect to $\bar{y}$

5. Conclusion

Table 4.1 clearly indicates that the class of suggested estimators is more efficient than classical ratio estimator, Singh and Vishwakarma (2007), Singh et al. (2008) and Noor-ul-Amin and Hanif (2012) estimators. It is also observed that among the class of suggested estimators, t_9 performs more efficiently in both of the cases. It is further observed that in Case-II the performance of the suggested estimators is better than in Case-I except for population-I. It is also concluded that deduced estimators t_{10}, t_{11} and t_{12} could be preferred over suggested estimator t_9 when the estimation of α is not possible in practice.

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