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ESTIMATION OF THE STATIONARY DISTRIBUTION OF A SEMI-MARKOV CHAIN

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Abstract

This article is concerned with the estimation of the stationary distribution of a discrete-time semi-Markov process. After briefly presenting the discrete-time semi-Markov setting, we propose an estimator of the associated stationary distribution. The main results concern the asymptotic properties of this estimator, as the sample size becomes large. A numerical example illustrates the asymptotic properties of the estimators.

Key words semi-Markov chains, stationary distribution, nonparametric estimation, asymptotic properties.

1. Introduction

Semi-Markov processes and Markov renewal processes represent a class of stochastic processes that generalize the Markov and the renewal processes. As it is well known, for a discrete-time (respectively continuous-time) Markov process, the sojourn time in each state is geometrically (respectively exponentially) distributed. In the semi-Markov case, the sojourn time distribution can be any distribution on $\mathbb{N}^*$ (resp. $\mathbb{R}_+$). For this reason, the semi-Markov approach is much more suitable for applications, than the Markov one (see, e.g., [1], [2], [5], [6], [8]).

A quantity related to a semi-Markov process is the so-called limit distribution (assumed to exist), which describes the limit behavior of the process, when time becomes large (cf. Definition 4). In a certain sense, this is also the stationary distribution of the chain (see the discussion after Definition 4). Estimating the stationary distribution of a semi-Markov chain is an important question, at least for two reasons. Firstly, from a theoretical or applied point of view, one is always interested in the equilibrium behavior of a process (when this equilibrium exists). Secondly, when a certain phenomenon has started sufficiently far in the past, one can always consider that it has reached it's equilibrium behavior when we actually begin the observation. When this is the case, it is justified to consider that the stationary distribution is the initial distribution of the process. As this initial distribution appears in the computation of many quantities we are interested in (for example, when computing the reliability/survival function, the availability, the failure rate, the mean time to failure/repair), it is important to be able to estimate this stationary distribution and to find estimators that have nice asymptotic properties.

To conclude, the purpose of this paper is to estimate the stationary distribution of a discrete-time semi-Markov process and to investigate the asymptotic properties of this estimator, as the sample size becomes large. Similar results have already been obtained in [7] for a continuous-time semi-Markov process.

The present article is structured as follows: in the next section we briefly introduce the semi-Markov framework and give the necessary notation and definitions. In Section 3 we define the stationary distribution of a semi-Markov chain, propose empirical estimators of the mean sojourn times of the semi-Markov chain and of the stationary distribution of the so called embedded Markov chain. Consequently, we obtain the corresponding estimator of the stationary distribution of the semi-Markov chain. Section 4 is devoted to the asymptotic properties of the estimator of the stationary distribution of the semi-Markov chain, namely to the strong consistency and asymptotic normality. We illustrate the theoretical results by a numerical example.

2. Discrete-time Semi-Markov Framework

In this section we introduce the basic notation concerning a discrete-time semi-Markov model. We consider a random system with finite state space $E = \{1, \dots, s\}$, whose evolution in time is governed by a stochastic process $Z = (Z_k)_{k \in \mathbb{N}}$. Let us denote by $S = (S_n)_{n \in \mathbb{N}}$ the successive time points when state changes in $(Z_k)_{k \in \mathbb{N}}$ occur and by $J = (J_n)_{n \in \mathbb{N}}$ the successively visited states at these time points. Set also $X = (X_n)_{n \in \mathbb{N}^*}$ for the successive sojourn times in the visited states; thus, $X_n = S_n - S_{n-1}$, $n \in \mathbb{N}^*$. Figure 1 gives a representation of the evolution of the system. The relation between process Z and process J of the successively visited states is given by $Z_k = J_{N(k)}$, or, equivalently, $J_n = Z_{S_n}, n, k \in \mathbb{N}$, where $N(k) := \max\{n \in \mathbb{N} \mid S_n \leq k\}$ is the discrete-time counting process of the number of jumps in $[1, k] \subset \mathbb{N}$.

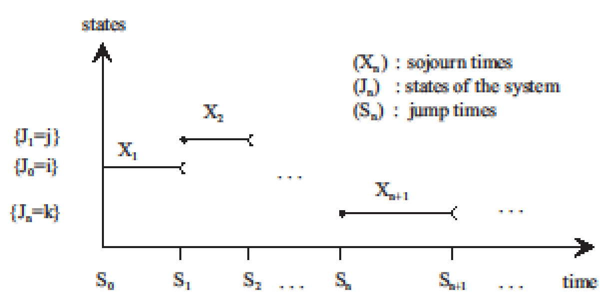

Figure 1: Sample path of a semi-Markov chain

We suppose that $Z = (Z_k)_{k \in \mathbb{N}}$ is a semi-Markov chain (SMC), or, equivalently, that the couple $(J, S) = (J_n, S_n)_{n \in \mathbb{N}}$ is a Markov renewal chain (MRC) and we denote by $\mathbf{q} = (q_{ij}(k); i, j \in E, k \in \mathbb{N})$ the associated discrete-time semi-Markov kernel defined by $q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} = k \mid J_n = i)$.

We also introduce the cumulative semi-Markov kernel $\mathbf{Q} = (\mathbf{Q}(k); k \in \mathbb{N})$ defined by

$$Q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} \leq k \mid J_n = i) = \sum_{l=0}^{k} q_{ij}(l), i, j \in E, k \in \mathbb{N} .$$

Note that, for (J, S) a Markov renewal chain, we can easily see that $(J_n)_{n \in \mathbb{N}}$, is a Markov chain, called *the embedded Markov chain (EMC) associated to the MRC* (J, S). We denote by $\mathbf{p} = (p_{ij})_{i, j \in E} \in \mathcal{M}_E$ the transition matrix of $(J_n)_{n \in \mathbb{N}}$,

$$p_{ij} = \mathbb{P}(J_{n+1} = j \mid J_n = i), i, j \in E, n \in \mathbb{N} .$$

Let the row vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ denote *the initial distribution* of the semi-Markov chain $Z = (Z_k)_{k \in \mathbb{N}}$, where $\alpha_i := \mathbb{P}(Z_0 = i) = \mathbb{P}(J_0 = i), i \in E.$

We also assume that $p_{ii} = 0$, $q_{ii}(k) = 0, k \in \mathbb{N}, i \in E.$ We define now the sojourn time distributions in a given state and the conditional distributions depending on the next state to be visited.

Definition 1 (Conditional distributions of the sojourn times) *For all* $i, j \in E$, *let us define:*

1. $f_{ij}(\cdot)$, *the conditional distribution of* $X_{n+1}, n \in \mathbb{N}$:

$$f_{ij}(k) = \mathbb{P}(X_{n+1} = k \mid J_n = i, J_{n+1} = j), k \in \mathbb{N}.$$

2. $F_{ij}(\cdot)$, *the conditional cumulative distribution of* $X_{n+1}, n \in \mathbb{N}$:

$$F_{ij}(k) = \mathbb{P}(X_{n+1} \leq k \mid J_n = i, J_{n+1} = j) = \sum_{l=0}^{k} f_{ij}(l), k \in \mathbb{N}.$$

Definition 2 (Sojourn times distributions in a given state) *For all* $i \in E$, *let us denote by:*

1. $h_i(\cdot)$, *the sojourn time distribution in state i:*

$$h_i(k) = \mathbb{P}(X_{n+1} = k \mid J_n = i) = \sum_{j \in E} q_{ij}(k), k \in \mathbb{N}^*.$$

2. $H_i(\cdot)$, *the sojourn time cumulative distribution function in state i:*

$$H_i(k) = \mathbb{P}(X_{n+1} \le k \mid J_n = i) = \sum_{l=1}^{k} h_i(l),\ k \in \mathbb{N}^*.$$

Let us also denote by m_i the mean sojourn time in a state $i \in E$,

$$m_j := \mathbb{E}(S_1 \mid J_0 = j) = \sum_{k \ge 1} k h_j(k) = \sum_{k \ge 1} (1 - H_j(k)).$$

Note that, for all $i, j \in E$ and $k \in \mathbb{N}$ such that $p_{ij} \neq 0$, the semi-Markov kernel verifies the relation

$$q_{ij}(k) = p_{ij} f_{ij}(k).$$

If we suppose that the sojourn times in a state depend only on the present visited state, a particular type of semi-Markov chain is obtained, whose semi-Markov kernel is $q_{ij}(k) = p_{ij} h_i(k),\ i, j \in E,\ k \in \mathbb{N}.$ For this particular type of semi-Markov chain we will prove the asymptotic normality of the stationary distribution estimator (Proposition 2).

For G the cumulative distribution function of a r.v. X, we denote its *survival function* by $\overline{G}(n) := 1 - G(n) = \mathbb{P}(X > n),\ n \in \mathbb{N}.$ Thus, for all states $i, j \in E$ we put $\overline{F}_{ij}$ and $\overline{H}_i$ for the corresponding survival functions.

Definition 3 *The transition function* $\mathbf{P} = (\mathbf{P}(k); k \in \mathbb{N})$ *of the semi-Markov chain Z is defined by*

$$P_{ij}(k) := \mathbb{P}(Z_k = j \mid Z_0 = i),\ i, j \in E,\ k \in \mathbb{N}.$$

3. Estimation of the Stationary Distribution

Definition 4 (limit distribution of a SMC) *For a semi-Markov chain* $(Z_k)_{k \in \mathbb{N}}$ *the limit distribution* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_s)^\top$ *is defined, when it exists, by* $\pi_j = \lim_{k \to \infty} P_{ij}(k),$ *for every* $i, j \in E.$

Let us denote by U_n the backward recurrence time $U_n := n - S_{N(n)}$ of the semi-Markov chain. It is worth noting that the limit distribution $\boldsymbol{\pi}$ is also the stationary distribution of the SMC $(Z_k)_{k \in \mathbb{N}}$ in the sense that it is the marginal distribution of the stationary distribution $\tilde{\pi}$ of the Markov chain $(Z_n, U_n)_{n \in \mathbb{N}}$, that is $\pi_j = \tilde{\pi}(\{j\}, \mathbb{N}),\ j \in E$ (see [1] & [3]). For these reasons, the limit distribution $\boldsymbol{\pi}$ will be also called the stationary distribution of the SMC.

All along this paper, we consider that the SMC Z is irreducible, aperiodic, with finite mean sojourn times.

Let $(S_n^j)_{n\in\mathbb{N}}$ be the successive passage times in a fixed state $j \in E$. For any arbitrary states $i, j \in E$, $i \neq j$, we denote by μ_{ij} the *mean first passage time from state i to j* for the SMC, $\mu_{ij} := \mathbb{E}_i(S_0^j)$, and by μ_{jj} the *mean recurrence time of state j* for the SMC, $\mu_{jj} := \mathbb{E}_j(S_1^j)$, where $\mathbb{E}_i$ is the conditional expectation given $\{J_0 = i\}$.

Proposition 1 *The limit distribution of an SMC is given by*

$$\pi_j = \frac{1}{\mu_{jj}} m_j = \frac{\nu(j)m_j}{\sum_{i\in E}\nu(i)m_i} = \frac{\nu(j)m_j}{\overline{m}}, \quad j \in E,$$

where the row vector $\mathbf{v} = (\nu(1),\ldots,\nu(s))$ *is the stationary distribution of the EMC* $(J_n)_{n\in\mathbb{N}}$ *and we denoted by* $\overline{m} := \sum_{i\in E}\nu(i)m_i$ *the mean sojourn time of the SMC.*

A proof of this result can be found in [1]. A different proof, based on generating functions, can be found in [4].

Let us assume now that we have an observation of this SMC, censored at fixed arbitrary time $M \in \mathbb{N}^*$, $(Z_0,\ldots,Z_M)$, or, equivalently, an observation of the associated Markov renewal chain $(J_n, S_n)_{n\in\mathbb{N}}$, $(J_0, X_1,\ldots,J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M)$, where $u_M := M - S_{N(M)}$ is the censored sojourn time in the last visited state $J_{N(M)}$.

For all states $i, j \in E$, let us introduce:

- $N_i(M) := \sum_{n=0}^{N(M)-1} \mathbf{1}_{\{J_n = i\}} = \sum_{n=0}^{M} \mathbf{1}_{\{J_n = i, S_{n+1}\le M\}}$ the number of visits to state i of the EMC $(J_n)_{n\in\mathbb{N}}$, up to time M;
- $N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n = j\}} = \sum_{n=1}^{M} \mathbf{1}_{\{J_{n-1}=i, J_n=j, S_n\le M\}}$ the number of transitions of the EMC $(J_n)_{n\in\mathbb{N}}$ from i to j, up to time M.

Let us consider the empirical estimator of the stationary distribution of the EMC $(J_n)_{n\in\mathbb{N}}$ defined by:

$$\hat{\nu}(i, M) = \frac{N_i(M)}{N(M)}, \; i \in E. \qquad (1)$$

For any state $i \in E$, writing the mean sojourn time in state i as $m_i = \sum_{k\ge 0}(1 - H_i(k)) = \sum_{k\ge 0}\overline{H}_i(k)$ and using the empirical estimator of the survival function in state i, $\overline{H}_i(k)$, we get an estimator for m_i,

$$\hat{m}_i(M) = \frac{1}{N_i(M)} \sum_{k=1}^{N_i(M)} X_{ik}. \tag{2}$$

Consequently, an estimator of the mean sojourn time of the SMC, $\overline{m}$, is

$$\hat{\overline{m}}(M) = \frac{1}{N(M)} \sum_{j \in E} \sum_{k=1}^{N_j(M)} X_{jk} = \frac{1}{N(M)} \sum_{k=1}^{N(M)} X_k, \tag{3}$$

and we get the following estimator of the stationary distribution of the SMC

$$\hat{\pi}_i(M) = \frac{1}{\hat{\overline{m}}(M) N(M)} \sum_{k=1}^{N_i(M)} X_{ik}, i \in E. \tag{4}$$

4. Asymptotic Results

First of all, note that we have the following asymptotic results:

$$N_i(M) / N(M) \xrightarrow[M \to \infty]{a.s.} \nu(i), \tag{5}$$

$$N_{ij}(M) / N(M) \xrightarrow[M \to \infty]{a.s.} \nu(i) p_{ij}, \tag{6}$$

$$N_i(M) / M \xrightarrow[M \to \infty]{a.s.} 1/\mu_{ii}. \tag{7}$$

The first two results are immediately obtained from classical Markov chain asymptotic properties, whereas the third one is a direct application of the Strong Law of Large Numbers (SLLN) to the simple renewal chain $(S_n^j - S_0^j)_{n \in \mathbb{N}}$.

Lemma 1 *For any state* $i \in E$ *of the SMC, the estimators* $\hat{\nu}(i, M)$, $\hat{m}_i(M)$, $\hat{\overline{m}}(M)$, *and* $\hat{\pi}_i(M)$ *proposed in Equations (1-4) for the stationary distribution of the EMC, mean sojourn time in state i mean sojourn time of the SMC, and stationary distribution of the SMC, respectively, are strongly consistent, as M tends to infinity.*

Proof.
The consistency of $\hat{\nu}(i, M)$ has been already stated in (5). From the SLLN and the fact that $N_i(M) \xrightarrow[M \to \infty]{a.s.} \infty$, we obtain the strong consistency of $\hat{m}_i(M)$ These results, together with continuous mapping theorem, yield the strong consistency of $\hat{\overline{m}}(M)$ and $\hat{\pi}_i(M)$, as M tends to infinity. □

The asymptotic normality of the stationary distribution estimator of a SMC will be proved for a particular semi-Markov model, defined by the semi-Markov kernel $q_{ij}(k) = p_{ij} h_i(k), \; i, j \in E, \; k \in \mathbb{N}$. Remark 1 given after the proof of the result gives details on this choice.

Proposition 2 *For any fixed arbitrary state* $i \in E$ *, we have*

$$\sqrt{M}[\hat{\pi}_i(M) - \pi_i] \xrightarrow[M \to \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2_{\pi_i}), \tag{8}$$

with asymptotic variance

$$\sigma^2_{\pi_i} = \frac{1}{\mu_{ii}} \frac{\dfrac{\sigma_i^2}{m_i^2} + \dfrac{\rho_{ii}^2 - \sigma_i^2}{(\mu_{ii} - m_i)^2}}{\left(\dfrac{1}{m_i} + \dfrac{1}{\mu_{ii} - m_i} \right)^2}, \tag{9}$$

where ρ_{ii}^2 *is the variance of the recurrence time of state i and* σ_i^2 *is the variance of the sojourn time in state i.*

Proof. The proof is essentially based on the limiting distribution of the total sojourn time in a state of semi-Markov process (cf. Theorem 3.1 of [9]).

Without loss of generality, for any fixed arbitrary state *i*, we can consider the initial visited state J_0 to be *i*. First, let us denote by $S_i(M)$ the total time spent by the SMC in state *i* up to time *M*, without taking into account the last censored time $u_M = M - S_{N(M)}$, i.e.,

$$S_i(M) = \sum_{i=1}^{N_i(M)} X_{ik} = \hat{m}_i(M) N_i(M)$$

and by $S_i^*(M)$ the total time spent by the SMC in state *i* up to time *M*, taking into account the last censored time, i.e.,

$$S_i^*(M) = S_i(M) + u_M \mathbf{1}_{\{J_{N(M)} = i\}}.$$

Second, let us express the variable of interest as follows:

$$\sqrt{M}[\hat{\pi}_i(M) - \pi_i] = \sqrt{M}\left[\frac{S_i(M)}{M - u_M} - \pi_i\right] = \sqrt{M}\left[\frac{S_i(M)/M}{1 - u_M / M} - \pi_i\right]$$

$$= \sqrt{M}\left[\frac{S_i(M)}{M} - \pi_i\right] + \frac{S_i(M)}{M}\frac{u_M}{\sqrt{M}}(1 + o_p(1))$$

$$= \sqrt{M}\left[\frac{S_i^*(M)}{M} - \pi_i\right] + \frac{S_i(M)}{M}\frac{u_M}{\sqrt{M}}(1 + o_p(1)) - \frac{u_M}{\sqrt{M}}\mathbf{1}_{\{J_{N(M)} = i\}}.$$

As $u_M / \sqrt{M} \xrightarrow[M \to \infty]{a.s.} 0$ and $\dfrac{S_i(M)}{M} = \hat{m}_i(M)\dfrac{N_i(M)}{M} \xrightarrow[M \to \infty]{a.s.} \dfrac{m_i}{\mu_{ii}} < \infty$, we get from Slutsky's theorem that $\sqrt{M}[\hat{\pi}_i(M) - \pi_i]$ has the same limit in distribution as

$\sqrt{M}\left[\frac{S_i^*(M)}{M}-\pi_i\right]$. Consequently, applying the result from [9] on the limiting distribution of the total sojourn time in a state of semi-Markov process, we get the desired result. □

Remark 1 *The proof of Theorem 3.1 of [9] is based on Takács's paper [10], that considers an alternating renewal chain* $(V_n)_{n\in\mathbb{N}^*}$, *with* $V_n := X_n + Y_n$, $n \in \mathbb{N}^*$, *under the assumptions that* $(X_n)_{n\in\mathbb{N}^*}$ *and* $(Y_n)_{n\in\mathbb{N}^*}$ *are sequences of i.i.d. random variables, and* $(X_n)_{n\in\mathbb{N}^*}$ *and* $(Y_n)_{n\in\mathbb{N}^*}$ *are independent between them. In order to apply this result in our framework, we need to consider sojourn times in a state depending only on the present visited state, i.e., a semi-Markov kernel of the form* $q_{ij}(k) = p_{ij}h_i(k)$.

The following result illustrates how the variance of the recurrence times can be recursively computed. A proof of this result could be found in [4].

Lemma 2 *For any state j, the variance* ρ_{jj}^2 *of the recurrence time of state j is given by*

$$\rho_{jj}^2 = \frac{1}{\nu(j)}\left[\sum_i \nu(i)(\sigma_i^2 + m_i^2) + \sum_i \sum_{k\neq j} 2m_i\mu_{kj}\nu(i)p_{ik}\right] - \mu_{jj}^2,$$

where the mean first passage times $\mu_{ij}, i, j \in E$, *can be computed using the following recurrence formulas (see [4] or[1] for a proof)*

$$\mu_{ij} = m_i + \sum_{k\neq j} p_{ik}\mu_{kj}, i, j \in E.$$

5. Numerical Example

In the following we demonstrate the findings of the previous section by means of a short simulation study. More precisely, we chose a 3-state SMC with shifted Poisson sojourn time distributions, that is,

$$h_i(k) = \frac{\lambda_i^{k-1}}{(k-1)!}e^{-\lambda_i}.$$

The true parameter values of the model equal

$$\mathbf{p} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.7 & 0 & 0.3 \\ 0.8 & 0.2 & 0 \end{pmatrix} \quad \text{and} \quad \lambda = \begin{pmatrix} 4 & 5 & 3 \end{pmatrix}.$$

Additionally, a uniform distribution is assumed for the initial distribution α. From this parameterization directly follows $m = \begin{pmatrix} 5 & 6 & 4 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 4 & 5 & 3 \end{pmatrix}$, $\nu = \begin{pmatrix} 0.429 & 0.274 & 0.297 \end{pmatrix}$, $\mu = \begin{pmatrix} 11.6 & 18.2 & 16.8 \end{pmatrix}$, and $\pi = \begin{pmatrix} 0.431 & 0.330 & 0.239 \end{pmatrix}$.

Thus, the true values of μ and π are available for checking the consistency of the estimators presented in Equation (7) and (4). Therefore, we simulate 200 sequences with *N(M) = 500* each, which is equivalent to values of *M* moderately superior to 2000.

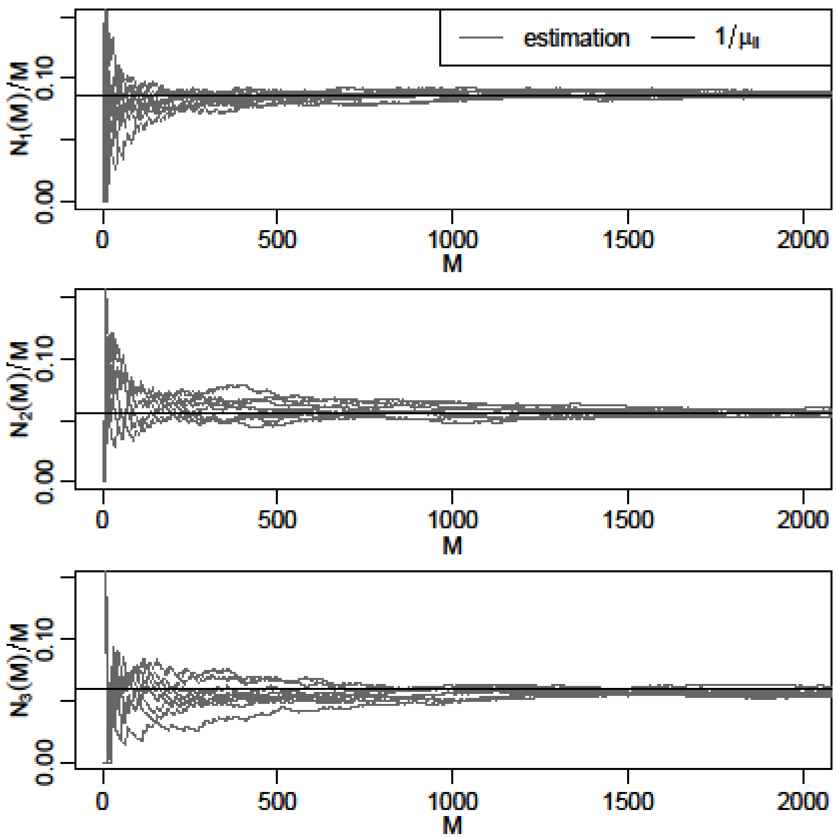

Figure 2: Estimated values of $N_i(M)/M$ (gray lines) from simulated series together with the true value of $1/\mu_{ii}$ (black lines) for states i=1, 2, and 3

Figures 2 & 3 provide a visual impression of convergence towards the true parameter values by means of 20 randomly selected sample paths. While the black horizontal lines represent the true values of $1/\mu$ and π, respectively, the gray lines result from the corresponding estimators. To confirm the optical impression, we calculate the empirical 5%- and 95%-quantile of the two estimators for *N(M) = 50*, *200* and *500*, respectively, from the 200 simulated trajectories. Table 1 displays the results, showing that the estimated quantities converge toward the true values for increasing *N(M)* (or *M*).

Finally, we investigate the findings of Proposition 2. Note that Equation (9) requires the calculation of ρ^2, the variance of the recurrence time of all states. Using a parametric bootstrap approach, we obtain the estimate $\hat{\rho}^2 = (20.8 \quad 108 \quad 122)$. Using this estimate and the true values of the remaining quantities described above, we obtain the "true" $\sigma_\pi^2 = (0.382 \quad 0.743 \quad 0.508)$. In order to check for normality of $\hat{\pi}$, we carry out the Shapiro-Wilk test. Furthermore, a potential dependence on M (*N(M)*) is investigated, we consider each 200 estimates $\hat{\pi}$ obtained for fixed $N(M) = 1, 2, \ldots, 500$ and test for normality by state. The results show that normality is rejected in a majority of cases for $N(M) \le 10$, sometimes for $10 < N(M) \le 50$, and rarely for $N(M) > 50$. Moreover, for increasing *M* the variance of the quantity on the left hand side of equation (8) converges towards the "true" values of σ_π^2. Figure 4 displays the evolution of variance of this quantity for increasing values of *N(M)* (or *M*). Recall that 200 observations serve for the variance estimation for each value of *N(M).*

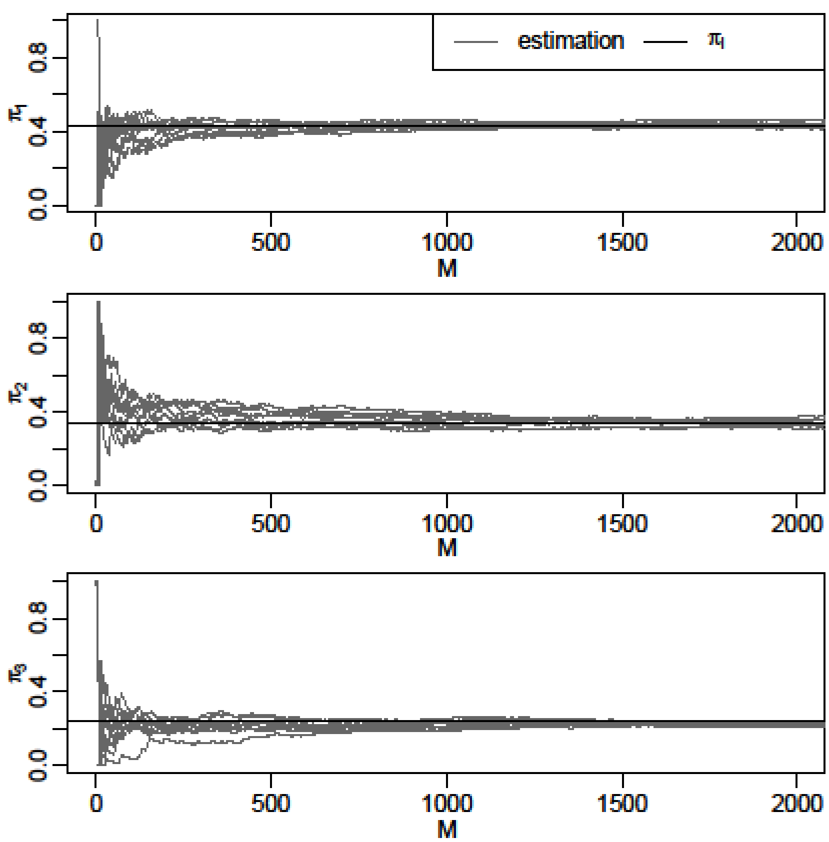

Figure 3: Estimated values of $\hat{\pi}_i(M)$ (gray lines) from simulated series together with the true value of π_i (black lines) for states *i*=1, 2, and 3

Table 1: Simulated results: 5%-and 95% quantile of μ & π for *N(M)*=50, 100, 500

		N(M)=50		*N(M)=200*		*N(M)=500*	
	true	q_{05}	q_{95}	q_{05}	q_{95}	q_{05}	q_{95}
$1/\mu_{11}$	0.0862	0.0747	0.098	0.0809	0.0921	0.0825	0.0899
$1/\mu_{22}$	0.055	0.0396	0.0691	0.0491	0.062	0.0509	0.0597
$1/\mu_{33}$	0.0596	0.0393	0.076	0.0505	0.0686	0.0544	0.0651
π_1	0.431	0.363	0.498	0.399	0.463	0.412	0.451
π_2	0.33	0.232	0.419	0.286	0.377	0.305	0.361
π_3	0.239	0.157	0.312	0.197	0.282	0.217	0.263

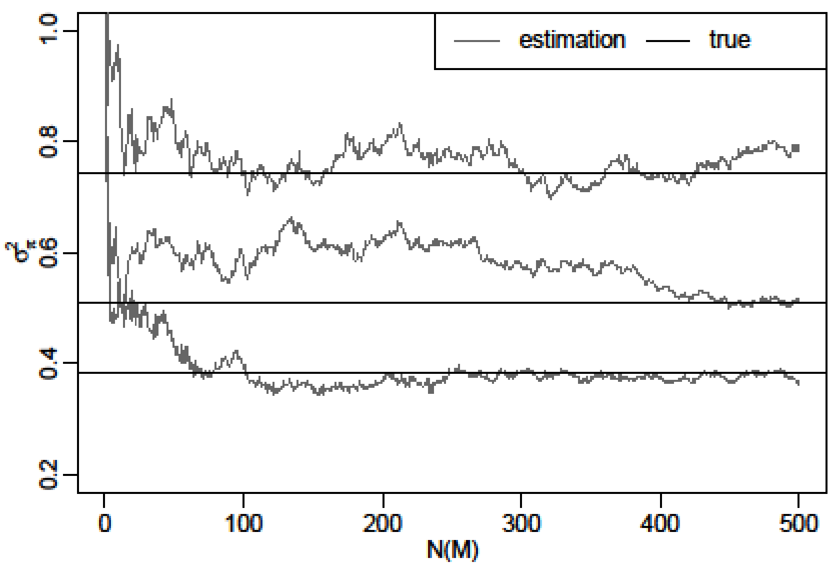

Figure 4: Estimated value of $\sigma^2_{\pi_i}$ (gray lines) from simulated series together with the true values (black lines) for states *i*=1, 2, and 3

Similar statements to those articulated above in the context of tests for normality hold true: For small values of *N(M)*, the sample variance is not too close to the target value, but this quickly changes for increasing *N(M).*

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