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AGEING AND RISK ASPECTS IN PREDICTIVE INFERENCE BASED ON PROPORTIONAL HAZARD MODELS

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Abstract

Proportional Hazard Models arise from a straightforward generalization of the simple case of conditionally i.i.d., exponentially distributed random variables and, in a sense, can be considered as the idealized models in the statistical analysis of failure and survival data for lifetimes. For these reasons, they have been extensively studied in the literature. Despite of the richness of related contributions, there are still special aspects of these models that are worthwhile focusing. In this discussion paper we aim to present some contributions, in the frame of a Bayesian approach and by using some very basic notions of stochastic ordering.

Key words: Archimedean copulas, Conditional IFR and DFR, Hazard rate ordering, Likelihood ratio, Majorization, More PQD, Stochastic ordering of posterior distributions, Two-actions decision problems.

1. Introduction

Generally speaking, we consider here some specific aspects of the problem of *predictive inference* for vectors of non-negative random variables $T_1,...,T_n$. We denote by $\bar{F}(t_1,...,t_n)$ the joint survival function of $T_1,...,T_n$, i.e. we put

$$\bar{F}(t_1,...,t_n) := P(T_1 > t_1,...,T_n > t_n).$$

Furthermore, we assume $\bar{F}$ to admit a regular joint density, to be denoted by $f(t_1,...,t_n)$. The random variables $T_1,...,T_n$ can have the meaning of *waiting times* or *time-durations* (of industrial components, living organisms, companies in a market, ...) and we then think of applications to different fields such as Reliability, Survival Analysis, Finance, Waiting Systems and so on. For our purposes, it will be convenient however to use the special language of Reliability Theory. In the related statistical analysis, one typically encounters statistical observations of the form

$$D \equiv \{T_{j_1} = t_1, T_{j_2} = t_2,...,T_{j_k} = t_k; T_{i_1} > s_1,...,T_{i_{n-k}} > s_{n-k}\}, \tag{1}$$

where $0 \le k \le n$, $0 \le t_1 \le .... \le t_k$, $0 \le s_1 \le ... \le s_{n-k}$. k is an observed *number of failures*, $t_1,...,t_k$ are the corresponding *failure times*, $s_1,...,s_{n-k}$ are *ages* or *survival times*, and $T_{i_1} - s_1$, ..., $T_{i_{n-k}} - s_{n-k}$ are *residual lifetimes*. We admit that $s_h = 0$, for some $h = 1,...,n-k$; in this case the variable T_{i_h} may not appear explicitly within the event *D*.

The term *predictive inference* conveys the idea that, putting ourselves in a frame of Bayesian Statistics, we are interested in inference about residual lifetimes, given observed data of the form (1). Namely we look at the joint conditional distribution of $T_{i_1} - s_1$, ..., $T_{i_{n-k}} - s_{n-k}$, given *D*. More precisely

a) we check *stochastic comparisons*, for a given *D*, between the marginal conditional distributions of $T_{i_{h'}} - s_{h'}$ and of $T_{i_{h''}} - s_{h''}$ respectively, for two different indexes $1 \leq h' < h'' \leq n-k$;

b) for a fixed index $1 \leq h \leq n-k$ and for special pairs of observations D' and D'', we establish stochastic comparisons between the conditional distributions of $T_{i_h} - s_h$, given D' and D'' respectively;

c) we compare stochastic dependence for the joint distributions of residual lifetimes, conditionally on D' and D'' respectively.

Of course, significant results in this direction can only be obtained by specifying some relevant probability models, i.e. special forms of the joint survival function $\overline{F}(t_1,...,t_n)$. In this paper we specifically consider the case of *Proportional Hazard Models* (often shortened, in the following, with P.H.M.). In this respect, we present some results that can be obtained in the afore-mentioned directions a), b) and c). As it is well known, the class of P.H.M.'s is a very special and interesting one, characterized by remarkable properties of both probabilistic and statistical type. For these reasons, they have been analyzed in the literature many times and from several different points of view. In particular, due to their properties, they emerge in a natural way in the statistical analysis of life-times data (see in particular [2, 15]). Actually they can be seen as very special cases of Cox Models, which in turn, find a natural generalization to the frailty models (see e.g. the recent review paper [28]). On the other hand, P.H.M.'s are strictly related with the popular topic of Archimedean copulas (see e.g. [20]). For a partial list of further relevant references see [1, 5, 18, 19].

In this paper we point out some unexplored aspects of Proportional Hazard Models and highlight some interesting relations existing among different properties that emerge in their statistical analysis. Such properties are related with concepts of risk and of *ageing*. Some of our results had already been presented, under a more general form, in [25, Ch. 5]. Notwithstanding this, and even though P.H.M.'s are very popular, still our discussion can shed a new light on the interest and on the meaning of the assumption that survival models are of this special type.

In order to recall the definition of P.H.M. we consider a *n*-tuple of non-negative, exchangeable, random variables $T_1,...,T_n$, to be thought of as lifetimes of similar units $U_1,...,U_n$. We talk of P.H.M. when the joint survival function $\overline{F}(t_1,...,t_n)$ has the form

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$$\overline{F}\left(t_1,...,t_n\right)=\int_0^{\infty}\exp\left\{-\theta\sum_{j=1}^{n}R(t_j)\right\}d\nu(\theta), \tag{2}$$

where $R:[0,+\infty)\to[0,+\infty)$ is a strictly increasing function such that $R(0)=0$. In other words, $T_1,...,T_n$ are conditionally i.i.d. given a *random parameter* Θ, taking values on $\left(0,+\infty\right)$, and ν can be seen, from a Bayesian view-point, as the *a-priori* distribution of Θ. These models arise as a straightforward generalization of the case of conditionally i.i.d. exponential lifetimes, that is characterized by the position

$$R(t)=t, \forall t>0. \tag{3}$$

The latter case can be considered as the idealized model in the analysis of life-times, in view of its, symmetry-related, properties. The latter properties can be seen as the statistical counterparts of the *memory-less property* of the exponential distribution. In our setting, they constitute the starting points whence natural questions arise, concerning the case (2). The attempt to give responses to such questions leads us to our developments here.

From a notational point of view notice that when, as in the present case, $T_1,...,T_n$ are exchangeable, it is not restrictive to give to (1) the simplified form

$$D\equiv\{T_1=t_1,T_2=t_2,...,T_k=t_k;T_{k+1}>s_1,...,T_{n-k}>s_{n-k}\}, \tag{4}$$

with $0<t_1\le...\le t_k$, $0\le s_1\le...\le s_{n-k}$.

The plan of the paper is as follows. In the next Section we review basic features of the P.H.M.'s that can be of greater interest for our purposes. Section 3 will start by pointing out the afore-mentioned properties for the cases when (3) holds and consequent questions concerning the more general P.H.M.'s. Then we present some results concerning conditional distributions of residual lifetimes; a subsection will be devoted to the special case defined by prior distributions of type gamma. A brief discussion concerning such results, and relations among them, will conclude the paper in Section 4.

Our language will require just a very basic knowledge of the fundamental univariate notions of ageing, such as *IFR*, *DFR*, *NBU*, etc, and of stochastic orders, such as the ordering *in the usual stochastic sense*, *in the hazard rate* and in *the likelihood ratio*. The latter will be respectively denoted, as usual, by $\le_{st}$, $\le_{hr}$, $\le_{lr}$. If needed, the reader can refer e.g. to [22]; see also [25, Ch. 3].

2. Some Relevant Properties of P.H.M.'s

As mentioned, the Proportional Hazard Models manifest numerous special properties that can be seen under different viewpoints and described in terms of different languages. In this Section we review those properties that have a major impact on our discussion. It is shown by (2) that a P.H.M., once its dimension *n* has been fixed, is characterized by the pair $\left(R,\nu\right)$. In the following we assume that the function R is differentiable and let $r(t) := dR(t)/dt$. This assumption allows $\bar{F}\left(t_1,...,t_n\right)$ to admit a regular joint density function, expressed by

$$f\left(t_1,...,t_n\right) = \prod_{j=1}^{n} r(t_j)\int_0^\infty \theta^n \exp\left\{-\theta\sum_{j=1}^{n} R(t_j)\right\} d\nu(\theta), \tag{5}$$

and, given $\Theta = \theta$, $T_1,...,T_n$ are conditionally i.i.d. with an absolutely continuous distribution, with $\theta r(t)$ as the univariate *hazard rate function*. The univariate marginal density of T_j ($j = 1,...,n$) is then given by

$$g(t) = r(t)\int_0^\infty \theta \exp\{-\theta R(t)\} d\nu(\theta). \tag{6}$$

In such an absolutely continuous case, we can easily deal with the conditional (or *posterior*) distribution ν_D of Θ given *D*, where *D* is the observation described in (4). As a function of θ, the *likelihood* of *D* is given by

$$L(\theta) = \theta^k \prod_{j=1}^{k} r(t_j) \exp\left\{-\theta\sum_{j=1}^{k} R(t_j)\right\} \exp\left\{-\theta\sum_{h=1}^{n-k} R(s_h)\right\}. \tag{7}$$

The latter equation has some important consequences that we list after the following technical remark.

Remark 1*: In the case when D contains some failure data, i.e. when k in (4) is greater than 0, the assumption that R is differentiable is important. In fact it allows us to construct the posterior distribution* ν_D *simply by means of the common Bayes Formula. On the contrary, if R is not differentiable and then the joint distribution of* $T_1,...,T_n$ *does not admit a probability density function, more delicate probability tools are needed in order to identify the conditional distribution of* Θ *given D. When, however, k=0 (i.e. when we only condition upon survival data), we can anyway compute* ν_D *by means of the common Bayes Formula even if the joint distribution of* $T_1,...,T_n$ *does not admit a probability density function.*

We come now back to listing the main consequences of (7). First of all, by suitably applying the Bayes Formula, we get that the distribution ν_D is expressed by

$$d\nu_D\left(\theta\right) = \frac{\theta^k \prod_{j=1}^{k} r(t_j) \exp\{-\theta\sum_{j=1}^{k} R(t_j)\} \exp\{-\theta\sum_{h=1}^{n-k} R(s_h)\} d\nu(\theta)}{\int_0^\infty \theta^k \prod_{j=1}^{k} r(t_j) \exp\{-\theta\sum_{j=1}^{k} R(t_j)\} \exp\{-\theta\sum_{h=1}^{n-k} R(s_h)\} d\nu(\theta)} =$$

$$= \frac{\theta^k \exp\{-\theta\sum_{j=1}^{k} R(t_j)\} \exp\{-\theta\sum_{h=1}^{n-k} R(s_h)\} d\nu(\theta)}{\int_0^\infty \theta^k \exp\{-\theta\sum_{j=1}^{k} R(t_j)\} \exp\{-\theta\sum_{h=1}^{n-k} R(s_h)\} d\nu(\theta)} \tag{8}$$

Let us now consider the residual lifetimes $T_1^{'},...,T_{n-k}^{'}$, defined by

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$$T_1' = T_{k+1} - s_1, ..., T_{n-k}' = T_n - s_{n-k}. \quad (9)$$

In view of conditional independence of $T_1,...,T_n$, given Θ, we can write easily the conditional density of $T_1',...,T_{n-k}'$, given *D*. In fact, $T_1',...,T_{n-k}'$ remain conditionally independent as well, given Θ; marginally, the conditional density of T_h' $(h=1,...,n-k)$, given both Θ and *D* is

$$g_{T_h'}(t \mid \theta, D) = \theta r(t+s_h)\exp\{-\theta\left[R(t+s_h)-R(s_h)\right]\}, t \geq 0, \quad (10)$$

and, as mixing distribution of Θ, we have the posterior ν_D. The marginal density of T_h' (conditional on *D*, but unconditional w.r.t. Θ) is then

$$g_{T_h'}(t \mid D) = r(t+s_h)\int_0^{+\infty} \theta \exp\{-\theta\left[R(t+s_h)-R(s_h)\right]\} d\nu_D(\theta). \quad (11)$$

In the special case when $s_1,...,s_{n-k} = s > t_k \geq ... \geq t_1 > 0$ (in such a case *D* is called a *history*), $T_1',...,T_{n-k}'$ are conditionally i.i.d., and then exchangeable. More precisely, they give rise to a P.H.M. of dimension $n' = n-k$, characterized by the pair (R_D, ν_D) where $R_D(t) = R(t+s) - R(s)$.

It clearly emerges from (7) and (8) that, whatever the prior distribution ν, the posterior distribution ν_D depends on *D* only through the pair (k,τ) where *k* denotes the observed number of failures and τ is the statistics defined by

$$\tau = \sum_{j=1}^{k} R(t_j) + \sum_{h=1}^{n-k} R(s_h).$$

Also the conditional distribution, given *D*, of the residual lifetimes depends only on (k,τ). In other words the pair (k,τ) is a *sufficient statistics*. It is a very remarkable property of these models the fact that a sufficient statistics of fixed dimension exists even for statistical observations that contain survival data. See the discussion in [2], see also [25].

The condition that $r(t)$ is monotonic reflects into remarkable properties of univariate, and multivariate, *ageing* for a P.H.M.. When $r(t)$ is decreasing, i.e. when conditionally on Θ, $T_1,...,T_n$ have a Decreasing Failure Rate (DFR) distribution, then also the marginal (i.e. *predictive*) distribution of $T_1,...,T_n$ is DFR, since (see e.g. [2]) any mixture of DFR distributions is DFR. Furthermore, the vector $(T_1,...,T_n)$ is also DFR, according to different multivariate definitions of DFR. In particular the corresponding joint survival function $\overline{F}(t_1,...,t_n)$ is *Schur-convex*, which can be seen as a property of multivariate DFR (see [25] and references cited therein). When $r(t)$ is increasing, $T_1,...,T_n$ have an Increasing Failure Rate (IFR) distribution conditionally on Θ. In this case we cannot ensure that the marginal (i.e. *predictive*) distribution of $T_1,...,T_n$ is IFR as well since, as it is well-known, a mixture of IFR distributions is not necessarily IFR. We still have, however, that $(T_1,...,T_n)$ has the multivariate IFR property described by $\bar{F}(t_1,...,t_n)$ being *Schur-concave*. We will come back to this multivariate ageing issue in Section 3.

In the study of qualitative properties of the conditional distribution of the residual lifetimes, given observed data of the form (4), the analysis of dependence properties of $T_1,...,T_n$ has clearly an important role. It is then an interesting issue looking at the *survival copula* $\hat{C}(u_1,...,u_n)$ of $T_1,...,T_n$. The latter (see e.g. [10, 20]) appropriately describes, in fact, the dependence properties of $T_1,...,T_n$ and is defined as the function $\hat{C}:[0,1]^n \to [0,1]$ given by

$$\hat{C}(u_1,...,u_n) = \bar{F}\left[\bar{G}^{-1}(u_1),...,\bar{G}^{-1}(u_n)\right],$$

where $\bar{G}$ denotes the survival function of the univariate marginal distribution, whose density has been given in (6). We notice that, by considering the Laplace transform

$$\tilde{W}(x) := \int_0^{+\infty} \exp\{-\theta x\} d\nu(x),$$

we can write

$$\bar{F}(t_1,...,t_n) = \tilde{W}\left(\sum_{j=1}^{n} R(t_j)\right), \tag{12}$$

whence

$$\bar{G}(t) = \tilde{W}(R(t)), \bar{G}^{-1}(u) = R^{-1}\left[\tilde{W}^{-1}(u)\right], 0 < u < 1, t \geq 0;$$

$$\hat{C}(u_1,...,u_n) = \tilde{W}\left[\ \tilde{W}^{-1}(u_1) + ... + \tilde{W}^{-1}(u_n)\right].$$

We then see that, in a P.H.M., $\hat{C}$ is not affected by the form of the function R and it is only determined by the distribution ν, through its Laplace transform. Furthermore, $\hat{C}$ is Archimedean and this can be seen as a further aspect of the special structure of P.H.M.'s. In view of (12), the class of P.H.M.'s can then be seen as a subset of the class of all the n-dimensional models characterized by survival functions of the form

$$\bar{F}(t_1,...,t_n) = W\left(\sum_{j=1}^{n} R(t_j)\right), \tag{13}$$

where $W:[0,+\infty) \to (0,1]$ is any decreasing function such that $\bar{F}$ in (13) is a *bona-fide* n-dimensional survival function (see in particular [16] for the analytic characterization of such a condition). On its turn, the condition (13) characterizes the class of all the continuous models with Archimedean survival copulas. Some of the arguments of this paper can be extended to such a class of models. However, we concentrate our attention on the case of P.H.M.'s (i.e. on the case of conditional i.i.d. lifetimes). Among all the multivariate models of the form (13), the latter ones emerge for their properties of positive dependence. Since, in a P.H.M., $T_1,...,T_n$ are conditionally i.i.d. with respect to the parameter Θ, and each of them is *stochastically decreasing* in Θ, one can heuristically expect, in fact, that they must satisfy some, weaker or stronger, property of

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positive dependence. Several detailed results along this direction have been obtained for the more general case of conditionally independent (not necessarily identically distributed) random variables and the related literature is well-known and very wide (see in particular [23, 12]). For the case of the P.H.M.'s we point out in this respect an important consequence of the fact that $\hat{C}$ is Archimedean: several properties of positive dependence, that are generally different one another, collapse into only one notion of positive dependence in the case of Archimedean copulas (see in particular [17, 6, 25]). This means that weak conditions of positive dependence actually reveal to be equivalent to much stronger ones.

3. Stochastic Comparisons for Residual Lifetimes Conditional on Total Time on Test Statistic

In order to introduce the topic of this Section we first consider the case of conditionally i.i.d. exponential lifetimes, which is characterized by joint survival functions of the form (2) with *R* satisfying (3). These models manifest obvious, but very remarkable, symmetry properties that can be briefly summarized as follows.

(a) Conditionally on any data *D* of the form in (4), the residual lifetimes $T'_{k+1},...,T'_n$ defined by (9) are identically distributed, even if $s_1,...,s_{n-k}$ are not all equal.

(b) For any data *D*, a sufficient statistic is provided by the pair $(k,\tilde{\tau})$, where, this time, $\tilde{\tau}$ coincides with the *Total Time on Test* statistics defined by

$$\tilde{\tau} = \sum_{j=1}^{k} t_j + \sum_{h=1}^{n-k} s_h.$$

Let us now consider, for the same vector of lifetimes $T_1,...,T_n$, two different sets of data *D* and D': *D* as in (4), and

$$D' \equiv \{T_1 = t'_1, T_2 = t'_2, ..., T_k = t'_k; T_{k+1} > s'_1, ..., T_n > s'_{n-k}\} \quad (14)$$

with $0 < t'_1 \le ... \le t'_k$, $0 \le s'_1 \le ... \le s'_{n-k}$. For simplicity of notation we in particular, assume the two vectors of survival times to be the same for *D* and D', i.e.

$$s'_1 = s_1, ..., s'_{n-k} = s_{n-k}.$$

We compare the conditional distributions for the residual lifetimes $T'_1,...,T'_{n-k}$ given *D* and D', respectively. Notice that we are assuming that both *D* and D' contain the same number of failures, k. We also assume that the two Total Time on Test statistics result in the same value, namely

$$\sum_{j=1}^{k} t_j = \sum_{j=1}^{k} t'_j. \quad (15)$$

For what it has been noticed above, we have that the conditional distributions of residual lifetimes share the same conditional distribution, irrespective of having observed D or $D^{'}$, since we originally assumed that $T_1,...,T_n$ are conditionally i.i.d., exponentially distributed. Such a property is extremely important in the statistical analysis of reliability data. Suppose for instance that we plan to collect some failure data in order to evaluate the reliability of a batch of similar components. Deciding the size of the sample to be tested is obviously part of the sampling procedure. Depending on different cost structures, sometimes we may want to collect in a short time a sufficient amount of Total Time on Test on the purpose of getting enough confidence about the quality. Some other times, concerning the same purpose, we have no problem about saving time; rather we have a limited number of items available to be tested. As a remarkable consequence of the assumption of conditional exponentiality, we reach the same state of information, and then of confidence about quality, from two different experiments, provided that we observe the same number k of failures and collect the same value τ for the TTT statistics (and provided that stopping of observation is *not informative*).

Of course we expect that the two conditional distributions of residual lifetimes, given D or $D^{'}$ respectively, would on the contrary be different, should the assumption of conditional exponentiality be removed. What can we then say in such a case? A general response is of course impossible, without specifying suitable properties for the joint density of $T_1,...,T_n$. Here we give some results under the condition that (5) holds with suitable properties of the function R and suitable majorization relations between the vectors $(t_1,...,t_k)$, $(t_1^{'},...,t_k^{'})$. To fix ideas, we assume that $(t_1,...,t_k)$, is *majorized* by $(t_1^{'},...,t_k^{'})$, namely (see [14])

$$t_1 \geq t_1^{'},\, t_1+t_2 \geq t_1^{'}+t_2^{'},\, ...,\, t_1+...+t_{n-1} \geq t_1^{'}+...+t_{n-1}^{'},\, \sum_{j=1}^{k} t_j = \sum_{j=1}^{k} t_j^{'} \,. \tag{16}$$

Proposition 2: Let (5) hold with R a convex function and ν an arbitrary probability distribution on $[0,\infty)$. Then

(a) for any $h = 1,...,n-k$,

$$T_h^{'} \mid D \leq_{lr} T_h^{'} \mid D^{'};$$

(b) for $1 \leq h^{'} < h^{''} \leq n-k$,

$$T_{h^{''}}^{'} \mid D \leq_{hr} T_{h^{'}}^{'} \mid D.$$

Proof: a) First we check that the posterior distributions ν_D and $\nu_{D^{'}}$ can be compared in the sense of the $\leq_{lr}$ stochastic ordering. On this purpose we consider the ratio $d\nu_D(\theta)/d\nu_{D^{'}}(\theta)$. By taking into account the expression of the type (8), for the distributions of Θ conditional on D and on $D^{'}$, respectively, we readily obtain in this respect that

$$\frac{d\nu_D(\theta)}{d\nu_{D^{'}}(\theta)} = \exp\left\{-\theta\left[\sum_{j=1}^{k} R(t_j) - \sum_{j=1}^{k} R(t_j^{'})\right]\right\}.$$

In view of the hypothesis (15) and of convexity of R, we obtain that this is an increasing function of θ, since

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$$\sum_{j=1}^{k} R(t_j) \leq \sum_{j=1}^{k} R(t_j^{'}).$$

The latter inequality can be seen e.g. by applying [14, Prop. 4.B.2]. We can now take into account that the conditional density (10) of $T_h^{'}$ given $1/\Theta = \eta$ is TP $_2$, i.e.

$$g(t^{'} \mid \eta^{'})g(t^{''} \mid \eta^{''}) \geq g(t^{''} \mid \eta^{'})g(t' \mid \eta^{''}).$$

As a direct consequence of the basic compositions formula of Totally Positive functions of order 2 (see [11]), we can conclude that the conditional densities of $T_h^{'}$ given D and $T_h^{'}$ given $D^{'}$ are ordered in the likelihood ratio ordering, as well (see also [25, Remark 3.16]).

b) For $1 \leq h^{'} < h^{''} \leq n-k$, we want to prove that the ratio

$$\rho(t) := \frac{\overline{F}_{T_{h^{'}}^{'}}(t \mid D)}{\overline{F}_{T_{h^{''}}^{'}}(t \mid D)}$$

is an increasing function of *t*. We recall the position $0 \leq s_1 \leq ... \leq s_{n-k}$. Now we have

$$\rho(t) = \frac{\int_0^{+\infty} \exp\{-\theta\left[R(t+s_{h^{'}}) - R(s_{h^{'}})\right]\} d\nu_D(\theta)}{\int_0^{+\infty} \exp\{-\theta\left[R(t+s_{h^{''}}) - R(s_{h^{''}})\right]\} d\nu_D(\theta)}.$$

By setting, for notational convenience,

$$d\hat{\nu}(\theta) = \frac{\exp\{-\theta\left[R(t+s_{h^{''}}) - R(s_{h^{''}})\right]\} d\nu_D(\theta)}{\int_0^{+\infty} \exp\{-\theta\left[R(t+s_{h^{''}}) - R(s_{h^{''}})\right]\} d\nu_D(\theta)},$$

and

$$Q(t) := R(t+s_{h^{'}}) - R(s_{h^{'}}) - R(t+s_{h^{''}}) + R(s_{h^{''}}), \tag{17}$$

we can write $\rho(t) = \int_0^{\infty} \exp\{-\theta Q(t)\} d\hat{\nu}(\theta)$. Now $\hat{\nu}(\theta)$ is a probability distribution over $[0,\infty)$ and, in view of the convexity assumption on *R*, $Q(t)$ is a decreasing function; therefore ρ is increasing in *t*.

The just proven stochastic orders change their direction if $R(t)$ is taken to be concave instead of convex.

Remark 3*: As explicitly said in the statement of Proposition 2, the validity of the two stochastic comparisons proven there is guaranteed by a suitable condition on the function R and is valid for any prior distribution* ν. *In particular, as shown in the proof,* ν *does not appear in the expression of the ratio* $d\nu_D(\theta)/d\nu_{D^{'}}(\theta)$.

The statement in b) of Proposition 2 can be strengthened by adding a suitable condition on the derivative of *R*.

Proposition 4: Let (5) hold with R convex, r log-concave and let ν be an arbitrary probability distribution on $[0,\infty)$. Then, for $1 \le h' < h'' \le n-k$,

$$T'_{h''} \mid D \le_{lr} T'_{h'} \mid D.$$

Proof: For $1 \le h' < h'' \le n-k$ we want to prove that the ratio

$$\rho(t) := \frac{r(t+s_{h'})\int_0^{+\infty} \theta \exp\{-\theta\left[R(t+s_{h'}) - R(s_{h'})\right]\} d\nu_D(\theta)}{r(t+s_{h''})\int_0^{+\infty} \theta \exp\{-\theta\left[R(t+s_{h''}) - R(s_{h''})\right]\} d\nu_D(\theta)}$$

is an increasing function of t. By setting

$$d\hat{\nu}(\theta) = \frac{\theta \exp\{-\theta\left[R(t+s_{h''}) - R(s_{h''})\right]\} d\nu_D(\theta)}{\int_0^{+\infty} \theta \exp\{-\theta\left[R(t+s_{h''}) - R(s_{h''})\right]\} d\nu_D(\theta)},$$

we can write

$$\rho(t) = \frac{r(t+s_{h'})}{r(t+s_{h''})}\int_0^{\infty} \exp\{-\theta Q(t)\} d\hat{\nu}(\theta)$$

where $Q(t)$ was defined in (17). Then $\int_0^{\infty} \exp\{-\theta Q(t)\} d\hat{\nu}(\theta)$ is increasing in t, since $Q(t)$ is decreasing and $r(t+s_{h'})/r(t+s_{h''})$ is increasing in t, since $r(t)$ is log-concave. Therefore $\rho(t)$ is increasing too.

Remark 5*: The statements in Proposition 2 and Proposition 4 are related to results presented in [4, 25]. Here we gave, for the special case defined by (2), specific statements and more direct proofs. In the case when $r(t)$ is increasing, we have the condition of Increasing Failure Rate for the lifetimes $T_1,...,T_n$, conditional on the parameter Θ. In such a case we also have that the joint survival function $\bar{F}$ is Schur--concave. When $r(t)$ is decreasing, we have for $T_1,...,T_n$ the condition of Decreasing Failure Rate, conditional on the parameter Θ and $\bar{F}$ is Schur-convex. The latter two properties of $\bar{F}$ can respectively be seen as conditions of multivariate IFR and multivariate DFR. Just as a rather immediate consequence of the definition of Schur-concavity, we can claim that $T'_{h''} \mid D \le_{st} T'_{h'} \mid D$ where D is as in the hypotheses of Proposition 2 (see e.g. the discussion in [25, Ch. 4] and references cited therein). Part b) of Proposition 2 achieved however the slightly stronger condition $T'_{h''} \mid D \le_{hr} T'_{h'} \mid D$. This result is made even stronger in Proposition 4.*

In the following result we obtain conclusions that are of a same type, but weaker than those in Proposition 2. On the other hand, we assume a condition on R that is weaker than convexity, namely we assume that it is *super-additive*, namely $R(x) + R(y) \le R(x+y)$. Furthermore we allow the vectors of survival times to be different, with a special relation between them; more precisely we take D' of the specific form.

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$$D' \equiv \{T_1 = t_1, T_2 = t_2, ..., T_k = t_k; T_{k+1} > s'_1, ..., T_n > s'_{n-k}\} \tag{18}$$

with $s'_1 = 0, ..., s'_\nu = 0$, and $s'_{\nu+1}, ..., s'_{n-k}$ with the following property: for a suitable partition $\mathcal{A} \equiv \{A_1, A_2, ..., A_{n-k-\nu}\}$ of the set of indexes $\{1, 2, ..., n-k\}$ we have

$$s'_{\nu+i} = \sum_{h \in A_i} s_h. \tag{19}$$

For simplicity of notation, we take the vectors of observed lifetimes equal in the two histories.

Proposition 6: Let (5) hold with R a super-additive function and ν an arbitrary probability distribution on $[0,\infty)$. Then

a) for any $h = 1, ..., n-k$,

$$T'_h \mid D \le_{lr} T'_h \mid D';$$

b) for $1 \le h' \le \nu$ and $\nu + 1 < h'' \le n-k$,

$$T'_{h''} \mid D' \le_{hr} T'_{h'} \mid D'.$$

Proof: a) Notice that, in view of (18), we can write

$$\frac{d\nu_D(\theta)}{d\nu_{D'}(\theta)} = \exp\left\{-\theta\left[\sum_{h=1}^{n-k} R(s_h) - \sum_{h=\nu}^{n-k} R(s'_h)\right]\right\}.$$

By recalling (19) and using super-additivity of R, we obtain that $d\nu_D(\theta) / d\nu_{D'}(\theta)$ is an increasing function of θ. From this point on, we can repeat the line of the proof of a) of Proposition 2.

b) We consider the ratio $\rho(t) := \bar{F}_{T'_{h'}}(t \mid D') \big/ \bar{F}_{T'_{h''}}(t \mid D')$ and proceed along the line of the proof of b) of Proposition 2.

We remark that, also for the statements of Proposition 6, one can repeat the considerations contained in Remark 3. We also point out that the super-additive condition on R means that the lifetimes are conditionally NBU, given Θ. Correspondingly, in view of the condition (19), the conclusion in b) can be seen as a sort of multivariate property of NBU, similarly to what was mentioned in Remark 5 about the cases of conditional IFR or conditional DFR.

In the next result, we come back to comparing two sets of data D and D' for which, as for Proposition 2, $(t_1, ..., t_k)$, $(t'_1, ..., t'_k)$ are different and $(t_1, ..., t_k)$, is majorized by $(t'_1, ..., t'_k)$. As to the survival times, we consider the case:

$$s_1 = ... = s_{n-k} = s'_1 = ... = s'_{n-k} = s. \tag{20}$$

We now look at the joint distributions of the residual lifetimes $T'_1, ..., T'_{n-k}$ conditional on D and D' respectively. In view of (20), both such distributions are exchangeable. Let us assume that R is convex. We then also know, from Proposition 2, that the univariate distribution of T_h' in the case D is dominated stochastically by the one corresponding to the case D'. We could wonder however which distribution may manifest a *stronger* positive dependence. Some insight is provided by the following Proposition 9, which will be obtained as a simple consequence of the lemma that we present next. On this purpose, it is convenient to introduce the following notation. Let π be a given probability density on $[0,+\infty)$ and let $\tilde{\pi},\hat{\pi}$ be two different densities of the form

$$\tilde{\pi}(\theta) \propto \theta^k \exp\{-\theta(a+t)\}\pi(\theta), \; \hat{\pi}(\theta) \propto \theta^k \exp\{-\theta(a+b+t)\}\pi(\theta).$$

Let us consider, furthermore, the Laplace transforms

$$\mathcal{L}(t) = \int_0^{+\infty} \theta^k \exp\{-\theta t\}\pi(\theta)d\theta, \tag{21}$$

$$\tilde{\mathcal{L}}(t) = \frac{1}{\mathcal{L}(a)}\int_0^{+\infty} \theta^k \exp\{-\theta(a+t)\}\pi(\theta)d\theta,$$

$$\hat{\mathcal{L}}(t) = \frac{1}{\mathcal{L}(a+b)}\int_0^{+\infty} \theta^k \exp\{-\theta(a+b+t)\}\pi(\theta)d\theta.$$

Notice that $\tilde{\pi},\hat{\pi}$ can be seen as the posterior densities that we obtain for the parameter Θ in a P.H.M. when we start from the same prior density $\pi(\theta)$, and observe two different sets of data. Each set of data contains the same number k of failures but possibly different vectors of failure times. However the following Lemma may be of a more general interest.

Lemma 7: If $b>0$, then $\hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(\cdot))$ is super-additive, i.e. for any $x, y \in \mathbb{R}_+$, $\hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(x)) + \hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(y)) \le \hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(x+y))$.

Proof: For any $x \in \mathbb{R}_+$, the equation $\hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(x)) = t$ is satisfied by $t \in \mathbb{R}_+$ such that $\tilde{\mathcal{L}}(x) = \hat{\mathcal{L}}(t)$, i.e.

$$\frac{1}{\mathcal{L}(a+b)}\int_0^{+\infty} \theta^k \exp\{\theta(a+b+t)\}\pi(\theta)d\theta = \frac{1}{\mathcal{L}(a)}\int_0^{+\infty} \theta^k \exp\{\theta(a+x)\}\pi(\theta)d\theta. \tag{22}$$

We notice now that both $\tilde{\mathcal{L}}$ and $\hat{\mathcal{L}}$ are strictly monotonic, as functions of the variable t. In view of this property, we can equivalently find the solution of (22) by equating the two integrands, i.e. by writing

$$\frac{1}{\mathcal{L}(a+b)}\exp\{-\theta(b+t)\} = \frac{1}{\mathcal{L}(a)}\exp\{-\theta x\}. \tag{23}$$

In fact, if we can find a t such that (23) is satisfied, it must be the only one t satisfying (22). From (23) we then obtain

$$t = \hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(x)) = x - b - \frac{1}{\theta}\log\left(\frac{\mathcal{L}(a+b)}{\mathcal{L}(a)}\right). \tag{24}$$

.

In view of the above equation, our thesis becomes

$$x-b-\frac{1}{\theta}\log\left(\frac{\mathcal{L}(a+b)}{\mathcal{L}(a)}\right)+y+-b-\frac{1}{\theta}\log\left(\frac{\mathcal{L}(a+b)}{\mathcal{L}(a)}\right)\le x+y+-b-\frac{1}{\theta}\log\left(\frac{\mathcal{L}(a+b)}{\mathcal{L}(a)}\right),$$

which is equivalent to

$$\frac{1}{\theta}\log\left(\frac{\mathcal{L}(a+b)}{\mathcal{L}(a)}\right)+b\ge 0 \ \text{ or } \ \exp\left\{\frac{1}{\theta}\log\left(\frac{\mathcal{L}(a+b)}{\mathcal{L}(a)}\right)+b\right\}\ge 1.$$

By recalling (21), the last inequality amounts to

$$\exp\{\theta b\}\int_0^{+\infty}\eta^k\exp\{-\eta a\}\pi(\eta)d\eta\ge\int_0^{+\infty}\eta^k\exp\{-\eta(a+b)\}\pi(\eta)d\eta.$$

Such an inequality trivially holds in view of the inequality $\exp\{-b(\theta+\eta)\}<1$ and of the monotonicity property of integrals.

Remark 8*: In practice, as a main point of the previous proof, we just shown that* $u(\cdot) := \hat{\mathcal{L}}^{-1}(\tilde{\mathcal{L}}(\cdot))$ *is an affine function, i.e. of the form* $u(t)=\gamma t+\delta$, *with* $\delta\ne 0$. *Notice that an affine function u is neither concave nor convex. Furthermore, for any* $\alpha\in[0,1]$ *and* $x,y\in\mathbb{R}_+$, *the following equality holds:*

$$\alpha u(x)+(1-\alpha)u(y)=u(\alpha x+(1-\alpha)y).$$

We now come back to comparing the conditional distributions of the residual lifetimes, given the two histories D and D'. Heuristically, we can expect that the more the probability distribution on the parameter Θ is dispersed, the riskier is any decision taken for the surviving units. On the other hand we can also guess that the more the probability distribution on Θ is dispersed, the stronger is the form of positive dependence among the residual lifetimes. In this standpoint, we wonder whether it is possible to order, according to some dependence ordering, the conditional distributions of the residual lifetimes corresponding to the observations of different sets of failure-data. Proposition 9 shows a simple result in this direction.

Let $(X_1', X_2'), (X_1'', X_2'')$ be two bivariate random vectors and let C' and C'' be their survival copulas. We can say that (X_1'', X_2'') is more positively dependent than (X_1', X_2'), in the PQD sense, if C' and C'' are ordered in the PQD sense (written $C' <_{PQD} C''$), namely if, for any $u_1,u_2\in[0,1]$, $C'(u_1,u_2)<C''(u_1,u_2)$.

For pairs of Archimedean copulas C_ϕ, C_ψ with generators φ,ψ respectively, the following result was proven in [1, Prop. 4]: the condition $C_\phi <_{PQD} C_\psi$ is equivalent to the condition that $\psi^{-1}(\phi(\cdot))$ is super-additive.

Let us come back now to considering the two histories D and D' as in (20) and the posterior distributions on Θ conditional on D and D', respectively ν_D and

$\nu_{D'}$ (see (8)). For simplicity of notation, let us suppose that the prior on Θ, ν, (and therefore its posteriors) admits a density π. The two densities π_D and $\pi_{D'}$, respectively of ν_D and $\nu_{D'}$, are given by

$$\pi_D(\theta) = \frac{\theta^k \exp\{-\theta \sum_{j=1}^{k} R(t_j)\} \exp\{-\theta(n-k)R(s)\}\pi(\theta)}{\int_0^\infty \theta^k \exp\{-\theta \sum_{j=1}^{k} R(t_j)\} \exp\{-\theta(n-k)R(s)\}\pi(\theta)d\theta},$$

$$\pi_{D'}(\theta) = \frac{\theta^k \exp\{-\theta \sum_{j=1}^{k} R(t_j')\} \exp\{-\theta(n-k)R(s)\}\pi(\theta)}{\int_0^\infty \theta^k \exp\{-\theta \sum_{j=1}^{k} R(t_j')\}\} \exp\{-\theta(n-k)R(s)\}\pi(\theta)d\theta}.$$

For shortness sake, we put

$$Q(s) := \sum_{j=1}^{k} R(t_j) + (n-k)R(s), \quad A(s) := \int_0^\infty \theta^k \exp\{-\theta Q(s)\}\pi(\theta)d\theta,$$

$$\hat{Q}(s) := \sum_{j=1}^{k} R(t') + (n-k)R(s), \quad \hat{A}(s) := \int_0^\infty \theta^k \exp\{-\theta \hat{Q}(s)\}\pi(\theta)d\theta,$$

so to write

$$\pi_D(\theta) = \frac{\theta^k \exp\{-\theta Q(s)\}\pi(\theta)}{A(s)}, \quad \pi_{D'}(\theta) = \frac{\theta^k \exp\{-\theta \hat{Q}(s)\}\pi(\theta)}{\hat{A}(s)}.$$

We recall that $\hat{Q}(s) > Q(s)$ and $\hat{A}(s) < A(s)$. We can write the Laplace transforms of ν_D and $\nu_{D'}$ as

$$W_D(t) = A(s)\int_0^\infty \theta^k \exp\{-\theta\left[Q(s)+t\right]\}\pi(\theta)d\theta,$$

$$W_{D'}(t) = \hat{A}(s)\int_0^\infty \theta^k \exp\{-\theta\left[\hat{Q}(s)+t\right]\}\pi(\theta)d\theta.$$

W_D and $W_{D'}$ can also be seen as the marginal univariate distributions of T_h ($h = 1,\dots,n-k$), conditional on D and D' respectively (and unconditional on Θ).

Let us consider now the bivariate Archimedean copulas C_{W_D} and $C_{W_{D'}}$ respectively generated by W_D and $W_{D'}$.

Proposition 9: Let $(t_1,...,t_k)$ be majorized by $(t_1',...,t_k')$. Then

$$C_{W_D} <_{PQD} C_{W_{D'}}.$$

.

Proof: We apply Lemma 7. By definition, we have $A(s) = \mathcal{L}(a)$ and $\hat{A}(s) = \mathcal{L}(a+b)$. Furthermore, we can set $a = Q(s)$ and $b = \sum_{j=1}^{k} R(t_j^{'}) - \sum_{j=1}^{k} R(t_j)$. This yields $\hat{Q}(s) = a + b$ and $W_D = \tilde{\mathcal{L}}, W_{D^{'}} = \hat{\mathcal{L}}.$

Since *b>0*, the hypotheses of Lemma 7 are satisfied and hence $W_{D^{'}}^{-1}(W_D(\cdot))$ is super-additive. We can then conclude the proof by resorting to the mentioned result shown in [1, Prop. 4], according to which the condition that $W_{D^{'}}^{-1}(W_D(\cdot))$ is super-additive is equivalent to $C_{W_D} <_{PQD} C_{W_{D^{'}}}$.

3.1 The Case of the Prior Distributions of Type Gamma

In a discussion dealing with models satisfying (5) one cannot omit mentioning the remarkable sub-class that is obtained by fixing the prior distribution ν in the family of gamma distributions. We devote this subsection to the discussion of some special aspects of this case. Let us then assume that the prior distribution ν admits a gamma density function with parameters α, β; namely we consider survival models with joint survival function given by

$$\bar{F}\left(t_1,...,t_n\right) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-1} \exp\left\{-\theta\left[\beta + \sum_{j=1}^{n} R(t_j)\right]\right\} d\theta = \frac{\beta^\alpha}{\left(\beta + \sum_{j=1}^{n} R(t_j)\right)^\alpha}. \qquad (25)$$

The corresponding one-dimensional marginal survival function is

$$\bar{G}(t) = \left(\frac{\beta}{\beta + R(t)}\right)^\alpha. \qquad (26)$$

The joint density function in (5) becomes

$$f\left(t_1,...,t_n\right) = \prod_{j=1}^{n} r(t_j) \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \frac{\beta^\alpha}{\left(\beta + \sum_{j=1}^{n} R(t_j)\right)^{n+\alpha}},$$

and the univariate marginal density of T_j ($j = 1,...,n$) is

$$g(t) = \alpha r(t) \frac{\beta^\alpha}{\left(\beta + R(t)\right)^{\alpha+1}}.$$

By adapting the equation (8), we easily see that the posterior distribution ν_D of Θ, given the observation *D* described in (4), is still a gamma distribution with the new parameters

$$\alpha^{'} = \alpha + k \ \ \& \ \ \beta^{'} = \beta + \sum_{j=1}^{k} R(t_j) + \sum_{h=1}^{n-k} R(s_h).$$

Concerning the residual lifetimes $T_1^{'},...,T_{n-k}^{'}$ defined in (9), the marginal density (conditional on D, but unconditional w.r.t. Θ) of $T_h^{'}$, $h=1,...,n-k$, correspondingly becomes

$$g_{T_h^{'}}(t)=r(t+s_h)\frac{(\beta')^{\alpha'}}{\Gamma(\alpha')}\int_0^{+\infty}\theta^{\alpha'}\exp\{-\theta[\beta^{'}+\left(R(t+s_h)-R(s_h)\right)]\}d\theta=$$

$$r(t+s_h)(\alpha+k)\frac{(\beta')^{\alpha+k}}{\left[\beta+\sum_{j=1}^{k}R(t_j)+\sum_{l=1,l\neq h}^{n-k}R(s_l)+R(t+s_h)\right]^{\alpha+k+1}}.$$

Taking into account that, for $\bar{G}$ as in (26), one has

$$\bar{G}^{-1}(u)=R^{-1}\left[\beta\left(\frac{1}{u^{1/\alpha-1}}\right)\right],$$

it is readily seen from (25) that the survival copula is an n-dimensional Archimedean Clayton copula with parameter α, that is $\hat{C}(u_1,...,u_n)=\left(1-n+u_1^{-\alpha}+...+u_n^{-\alpha}\right)^{-1/\alpha}$.

This shows explicitly that β and $R(t)$ have no part in determining the form of $\hat{C}$. We remarked above that the particular choice of ν has no influence on the results considered in Propositions 2, 4 & 6. A different situation is on the contrary met for what may concern the object of Proposition 9. Something interesting happens in this respect under the special choice of a gamma density. Let us consider in fact the joint distribution of the residual lifetimes $T_1^{'},...,T_{n-k}^{'}$ given a history $D\equiv\{T_1=t_1,...,T_k=t_k,T_{k+1}>s\}$ with $0\le t_1\le...\le t_k\le s$. $T_1^{'},...,T_{n-k}^{'}$ are conditionally i.i.d, given $\Theta=\theta$, with the conditional density

$$g_{T_h^{'}}(t\mid\theta)=\theta r(t+s)\exp\{-\theta\left[R(t+s)-R(s)\right]\}$$

and the conditional distribution of Θ given D is again gamma, with parameters

$$\alpha^{'}=\alpha+k,\ \ \beta^{'}=\sum_{j=1}^{k}R(t_j)+(n-k)R(s).$$

This means that the survival copula of the residual lifetimes remains Clayton with parameter $\alpha^{'}$, independently on the observed failure times $t_1\le...\le t_k$ (as far as k remains fixed) and on the value of the survival time (of surviving components) s. Then, the dependence structure of the vector of the residual lifetimes remains fixed and, in particular, we do not have any phenomenon of extreme tail dependence. Such a property, of truncation invariance, of Clayton copulas has been considered several times in the recent years, especially in the frame of financial risk (see in particular [27, 21, 7, 8]). In a Bayesian standpoint, we can look at this property as a direct consequence of the fact that the family of gamma distributions is *conjugate* to sampling from proportional hazard models, as seen above. This property can also be seen as one of a fixed-point type under the transformations given by time-truncation. Related with this stand-point, one can state that, under appropriate conditions, the limit of a P.H.M. conditional on survivals at time *s*, is just a model of the form (25).

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When ν is not of a gamma type, ν_D generally depends on the observed failure times $t_1,...,t_k$. It is then meaningful to wonder, among two different vectors of observed failure times, whether one of the two gives rise to a stronger form of positive dependence among the residual lifetimes, or to some form of tail dependence. A response in this direction is provided by Proposition 9.

4. Discussion and Concluding Remarks

In this section, we add some further remarks about the meaning of the results presented above and about related aspects of the assumption of P.H.M. To this purpose, we resort to the simple scheme of choices in a *two-actions decision problem*.

Suppose that, initially, we have *n* (apparently) identical units and let $T_1,...,T_n$ denote their lifetimes. $T_1,...,T_n$ are jointly distributed according to a P.H.M. with $r(t) = R^{'}(t)$ monotonic. Relatively to each unit, we should choose one between two different actions a_1 and a_2. The loss corresponding to the decision a_i ($i = 1,2$) for the unit *j* depends on the lifetime T_j and is quantified by $L_i\left(T_j\right)$, for two given loss functions $L_i : \left(0,\infty\right) \to \mathbb{R}$. In the choice between the two actions we hinge on the expected-loss minimization principle. Namely, we choose a_2 for the unit *j* if and only if the inequality $\mathbb{E}\left(L_2(T_j\right)) \le \mathbb{E}\left(L_1(T_j\right))$ holds. For simplicity sake we make the assumption that $L_2(t) - L_1(t)$ is a decreasing function of *t*. This entails that we can look at a_2 as the *more optimistic* decision, namely the one to be preferred should we knew that a unit life-time is large enough; a_1 would be then the *less optimistic* decision. Obviously, being $T_1,...,T_n$ exchangeable, and then in particular identically distributed, the same action has to be chosen for all *n* units.

Suppose now that the initial state of information is too scarce and then taking the decision is considered too risky, i.e. we judge $\min\left(\mathbb{E}\left(L_1(T_j)\right), \mathbb{E}\left(L_2(T_j)\right)\right)$ to be too large. Thus we defer the decisions to the evidence provided by some life-testing experiment (to be conducted according to some appropriate sample strategy) and we then collect some life-time data. Let data *D*, of the form (4), be actually collected. At this stage, *k* failures have been observed and the choice of decisions is limited only to the surviving units $h = 1,...,n-k$. We have to look at the distribution of the residual lifetimes $T_1^{'},..,T_{n-k}^{'}$, which are not necessarily exchangeable, anymore. Which is the rational decision to be taken for each of them? Denote by $I(D)$ the subset of the indexes *j* for which the inequality $\mathbb{E}\left(L_2(T_h^{'}) \mid D\right) \le \mathbb{E}\left(L_1(T_h^{'}) \mid D\right)$ holds. We recall, to fix ideas, that the survival times $s_1,...,s_{n-k}$ are arranged in the non-decreasing order. We analyze the set $I(D)$ and compare it with $I(D^{'})$ for a different set of data $D^{'}$.

Assume the hypotheses of Proposition 2; in particular $R(t)$ is convex ($r(t)$ is increasing). Then part a) of Proposition 2 says that $I(D) \subseteq I(D^{'})$, whereas part b) says that $I(D)$, when it is not empty, must be of the form $I(D) = \{1,\ldots,h_D\}$, for some index $h_D \in \{1,...,n-k\}$. Similarly $I(D^{'}) = \{1,\ldots,h_{D^{'}}\}$ (with $h_{D^{'}} \geq h_D$).

When, on the contrary, R is concave (namely a model of early failures is taken), then it must be $I(D^{'}) \subseteq I(D)$. But, this time, $I(D)$ and $I(D^{'})$ must be of the form $\{u_D,\ldots,n-k\}$, $\{u_{D^{'}},\ldots,n-k\}$ with $u_{D^{'}} \geq u_D$.

The above considerations lead us to conclude that, if we assume $r(t) = R^{'}(t)$ to be monotonic, the following situation must hold: no index $j^{'}$ belonging to $I(D^{'})$ can exist such that $j^{'} < j$ for every $j \in I(D)$. Suppose in other words that, once the data $D^{'}$ have been observed, it is rational to take the decision a_2 for a surviving unit with residual lifetime $T_j^{'}$ and age s_j. Then, it must exist some less aged surviving unit, for which it would be rational to take a_2, should the observation of D be taken.

We stress that the above conclusions are valid under the assumption that the failure rate function is monotonic, irrespective of assuming a model of positive ageing (r increasing) or of early failures (r decreasing).

Proposition 4 has mainly a technical purpose. By establishing a stronger form of stochastic comparison (namely $\leq_{lr}$ in place of $\leq_{hr}$), it allows us to replace the condition that $L_2 - L_1$ is decreasing with the following weaker one: $L_1 - L_2$ has only a change of sign, namely there exist $\overline{t}$ such that $\left[L_2(t) - L_1(t)\right](t - \overline{t}) < 0$ for any $t \neq \overline{t}$ (see e.g. the considerations in [25, Ch. 3].

Proposition 6 leads us to the same conclusion as Proposition 2; the difference between the two statements dwells in the circumstance that a weaker condition on R is required, and the two vectors of survival times, $\mathbf{s}$ and $\mathbf{s}^{'}$, are different. But the relation between $\mathbf{s}$ and $\mathbf{s}^{'}$ must be of a special form.

A rather different message is, on the contrary, contained in Proposition 9, where we compare two histories containing different vectors of failure times. Such a result shows that it is possible to compare (in the sense of PQD order) the strength of positive dependence between pair of residual lifetimes, given the two different histories.

All the above facts are in the general spirit of [9]; however they are specific of the P.H.M.'s and point out the special structure of such models. Indeed the inequalities we proved are not generally guaranteed if we consider conditionally i.i.d. IFR or conditionally i.i.d. DFR variables, outside of the P.H.M.'s. One related aspect is that a

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same type of inequalities can be obtained for all the P.H.M.'s. with a same concavity-convexity character of $R(t)$, irrespective of the prior distribution ν (see Remark 1).

In more general cases of conditional independence given a parameter Θ, on the contrary, we cannot exclude some interactions between the prior distribution on Θ and ageing properties of the joint model.

References

1. Averous J. and Dortet-Bernadet J. L. (2004). Dependence for Archimedean copulas and aging properties of their generating functions. Sankhya 66, p. 607--620.
2. Barlow R. E. and Proschan F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.
3. Barlow R. E. and Spizzichino F.(1993). Schur-concave survival functions and survival analysis. J. Comp. and Appl. Math. 46, p. 437--447.
4. Bassan B. and Spizzichino F. (1999). Stochastic comparison for residual lifetimes and Bayesian notions of multivariate ageing. Adv. In Appl. Probab. 31, p. 1078--1094.
5. Bassan B. and Spizzichino F. (2003). On some properties of dependence and aging for residual life-times in the exchangeable case. Mathematical and Statistical Methods in Reliability, World Scientific.
6. Caramellino L. and Spizzichino F. (1996). WBF property and stochastic monotonicity of the Markov Process associated to Schur-constant survival functions. J. Multiv. Anal, 56, p. 153--163.
7. Charpentier A. (2006). Dependence structure and limiting results: some applications in finance and insurance. PhD thesis, Katholieke Universiteit Leuven.
8. Durante F. and Jaworski P. (2010). Spatial contagion between financial markets: a copula-based approach. Appl. Stoch. Models Bus. Ind. 26, p. 551--564.
9. Fahmy S., de B. Pereira C. A., Proschan F. and Shaked M. (1982). The influence of the sample on the posterior distribution. Comm. Statist. - A. Theory and Methods, 11, p. 1757--1768.
10. Joe, H. (1997). Multivariate models and dependence concepts, Monographs on Statistics and Applied Probability, vol.73. Chapman & Hall, London.
11. Karlin S. and Rinott Y. (1980). Classes of orderings measures and related correlation inequalities. I, Multivariate totally positive distributions. J. Mult. An., 10, p. 467--498.
12. Khaledi B.E. and Kochar S. (2001). Dependence properties of multivariate mixture distributions and their applications. Ann. Inst. Statist. Math., 53, p. 620--630.
13. Lai C.D. and Xie M. (2006). Stochastic Ageing and Dependence for Reliability. Springer-Verlag, New York.
14. Marshall A. and Olkin (1979). Inequalities: Theory of Majorization and its Applications. Academic Press, New York.
15. Marshall A. and Olkin I. (1988). Families of Multivariate Distributions. J. Amer. Statist. Soc. 83, p. 834--841.

16. McNeil A. and Neŝhelová A. (2009). Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. Ann. Statist. 37, p. 3059--3097.
17. Mosler K. and Scarsini M. (1991). Some theory of stochastic dominance. In Stochastic Orders and Decision under Risk; Mosler and Scarsini eds., Lecture Notes-Monograph Series, 19, p. 261—284. Institute of Mathematical Statistics, Hayward, CA.
18. Mulero J. and Pellerey F. (2010). Bivariate Aging Properties under Archimedean Dependence Structures. Comm. Statist. - Theory and Methods, 39 (17), p. 3108--3121.
19. Müller A. and Scarsini M. (2005). Archimedean copulae and positive dependence. J. Multiv. Analysis, 93, p. 443--445.
20. Nelsen R. (2006). An introduction to copulas. 2nd ed., Springer Series in Statistics. New York, NY: Springer.
21. Oakes D. (2005). On the preservation of copula structure under truncation. Can. J. Stat. , 33 (3), p. 465--468.
22. Shaked M. and Shanthikumar J. G. (2007). Stochastic Orders and Their Applications, Springer Verlag.
23. Shaked M. and Spizzichino F. (1998). Positive dependence properties of conditionally independent random lifetimes. Math. Op. Res., 23, p. 944--959.
24. Spizzichino F.: Reliability decision problems under conditions of ageing. In: J. Bernardo, J. Berger J., Dawid, A.P. and Smith, A.F.M. (eds.) (1992). Bayesian Statistic, 4, 803--811, Clarendon Press, Oxford, UK.
25. Spizzichino F. (2001). Subjective probability models for life-times. Chapman and Hall/CRC, Boca Raton, Fl.
26. Spizzichino F. (2007). Ageing and positive dependence. In: F. Ruggeri, R. Kennett, F.W. Faltin (eds) Encyclopedia of Statistics for Quality and Reliability, p. 82-95, Wiley, Chichester, UK.
27. Sungur E. A. (2002). Some results on truncation dependence invariant class of copulas. Commun. Stat. - Theory and Methods 31 (8), p. 1399--1422.
28. Vonta F. (2012). Frailty or transformation models in survival analysis and reliability. In Recent Advances in System Reliability: Signatures, Multi-state Systems and Statistical Inference, Lisniansky, A. and Frenkel, I. (eds.), Springer, London, 237-252.