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INFERENTIAL ANALYSIS OF THE RE-MODELED STRESS-STRENGTH SYSTEM RELIABILITY WITH APPLICATION TO THE REAL DATA

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Abstract: The present study deals with the classical and Bayesian analysis of re-modeled stress-strength system reliability by considering Weibull distribution as the distribution of both the stress and strength variables. The proposed re-modeled stress-strength system reliability is defined as the probability that the system is capable to withstand the maximum operated stress at its minimum strength i.e., P[U>V], where $U=Min(X_1, X_2 \ldots X_m)$ and $V=Max(Y_1, Y_2, \ldots\ldots Y_n)$. The observations $X_1, X_2 \ldots X_m$ and $Y_1, Y_2, \ldots\ldots Y_n$ are the measurements on the strength and stress variables at different time epochs. The goodness-of-fit of the two real data sets for the proposed model is also demonstrated.

Keywords: Stress-Strength model, maximum likelihood estimate, Bayes estimate, empirical distribution function, Gibbs sampler, Metropolis-Hastings algorithm, highest posterior density credible interval.

1. Introduction

The stress-strength reliability model, R=P(X>Y), considers uncertainties about the actual environmental stress to be encountered as well as the resistance bearing properties of a component. Thus, the root of this model is that a component is always subjected to variety of stresses during its life-time and whether or not the component fails due to these stresses is just a matter of whether the resistance (strength) provided by the component is greater than the load (stress) applied. Here, the random variables X and Y respectively denote the strength and stress of the component. This stress-strength reliability model can be applied to numerous specific engineering cases, military problems and medical related problems of which clinical trials is one of the fastest growing areas. Some practical situations in which stress-strength model can be applied are as follows:

- *"Break-even voltage (X) of a capacitor must exceed the voltage output (Y) of a power supply for the proper functioning of the equipment."* [Krishnamoorthy, Mukherjee and Guo, 2006].
- *"If Y represents the diameter of a shaft and X represents the diameter of a bearing that is mounted on the shaft, then the 'R' is the probability that the bearing fits without interference."* [Nadarajah, 2005].

The field seems to have reached maturity as massive literature is devoted to probabilistic problems associated with the assessment of P(X>Y) and its estimation but the germ of this idea was originated in Birnbaum, 1956 and then Birnbaum and McCarty, 1958 extended it. They studied the point and interval estimation procedures of P(X>Y). By the late eighties, R=P[X>Y], a measure of system performance, was estimated for most of the statistical distributions for the scenario when X and Y are assumed to be independent of each other [Church and Harris, 1970; Kelley et al., 1976;

Tong, 1977; Beg and Singh, 1979; Voinov, 1984; Awad and Gharraf, 1986]. Recently, Baklizi and El-Masri, 2004 obtained the shrinkage estimator of R=P(X>Y) for two-parameter exponential distribution when X and Y are considered to be independent random variables. Further, Kundu and Gupta, 2005; Raqab and Kundu, 2005 and Krishnamoorthy, Mukherjee, and Guo, 2006 obtained the estimates of stress-strength reliability model for two-parameter generalized exponential, scaled Burr-type X and two-parameter exponential distributions respectively. With the introduction of bivariate distributions, it became possible to study inter-dependency between the stress and strength variables. Estimators of P(X>Y) for bivariate exponential random vector [Awad et al., 1981; Abu-Salih and Shamseldin, 1988; Jana and Roy, 1994], for bivariate normal random vector [Mukherjee and Saran, 1985], for bivariate gamma distribution [Nadarajah, 2005] have also been derived**.** In recent years, a few less familiar distributions like Wienman exponential, bivariate Pareto, multivariate normal and other are considered for drawing inferences on R=P(X>Y). Cramer and Kamps, 1997a and Cramer, 2001 estimated the component reliability considering the Wienman exponential distribution for stress and strength variables whereas Singh, 1981; Gupta and Gupta, 1990 considered multivariate normal distribution for obtaining the estimates for P(X>Y) and for drawing the inference on the same. In addition to the above studies, Bayesian analysis of stress-strength reliability model for normally distributed variables [Weerahandi and Johnson, 1992], for Burr-type X distribution [Ahmad, Fakhry and Jaheen, 1997; Kim and Chung, 2006] have also been conducted.

In all the above studies, the authors considered that the system is operable only when its strength exceeds the stress encountering at the time of operation. However, it is observed especially in military and medical sciences that the system's designers, reliability practitioners and experts in medical field seek to assign high probability to the event that the system/unit remains operable at its minimum strength encountering maximum stress at that time epoch. For example- in the defence combats, the warfare equipments should have very high reliability in order to perform its intended function of defence and attack satisfactorily. Similarly, another interesting example could be of stress fracture from the orthopedic surgery. Bones are flexible tissues capable of repair, and regeneration in response to both mechanical stresses and environmental strains or deformations. In normal conditions, bones are gifted to keep up with the necessary fixations without manifesting clinically significant injuries. However, whenever a bone's minimal reparative and adaptive ability (strength) is exceeded by chronic overstress, damage can begin to accumulate and fracture occurs.

Thus, to meet the above objective, it seems logical to re-modeled the stress-strength reliability as P [U>V], where $U=Min(X_1, X_2 \ldots X_m)$ and $V=Max(Y_1,Y_2,\ldots\ldots Y_n)$. The observations $X_1, X_2 \ldots X_m$ and $Y_1,Y_2,\ldots\ldots Y_n$ are the measurements on the strength and stress variables at different time epochs. Thus, the proposed re-modeled stress-strength system reliability is defined as the probability that the system is capable to withstand the maximum operated stress at its minimum strength.

Further, it is commonly observed that the failure rates of the components vary with time. Hence, it is reasonable and more pragmatic to assume Weibull distribution as the failure time distribution of the components/units. It is one of the most extensively used distribution in reliability theory as its analysis provides the information needed for classifying the failure types, scheduling preventive maintenance and scheduling inspection. Also, Weibull distribution can characterize verity of data because of its shape parameter.

In view of the above, the present study deals with the analysis of re-modeled stress-strength system reliability by considering Weibull distribution as the distribution of both the stress and strength variables. The assumptions are given in section 2. The expression for the re-modeled system reliability is derived in section 3. In section 4.1, we obtain maximum likelihood estimate of the re-modeled system reliability. Further, since, the life-testing experiments are very time consuming as such the parameters involved in the stress and strength distributions are assumed to be random variables. Therefore, re-modeled system reliability is also analyzed in Bayesian setup. The Markov Chain Monte Carlo (MCMC) technique such as Gibbs sampler has been utilized for this purpose. At the end, we demonstrate the goodness-of-fit of the two real data sets for the proposed model.

2. Statistical Assumptions

For developing the stress-strength reliability model, it is assumed that:

(i) The strength variable X is distributed with p.d.f.

$$f_1(x,\lambda,\beta_1)=\beta_1\lambda\, x^{\beta_1-1}e^{-\lambda x^{\beta_1}} \quad ,x\geq 0,\lambda>0,\beta_1>0 \tag{1}$$

(ii) The stress variable Y is distributed with p.d.f.

$$f_2(y,\theta,\beta_2)=\beta_2\,\theta\, y^{\beta_2-1}e^{-\theta y^{\beta_2}} \quad ,y\geq 0,\theta>0,\beta_2>0 \tag{2}$$

(iii) In view of (i), the p.d.f. of the random variable ‘U’ will be

$$g_1(u,\lambda,\beta_1)=m\beta_1\lambda\, u^{\beta_1-1}e^{-m\lambda u^{\beta_1}} \quad ,u\geq 0,\lambda>0,\beta_1>0 \tag{3}$$

(iv) In view of (ii), the p.d.f. of the random variable ‘V’ will be

$$g_2(v,\theta,\beta_2)=n\beta_2\theta\, v^{\beta_2-1}e^{-\theta v^{\beta_2}}(1-e^{-\theta v^{\beta_2}})^{n-1} \quad ,v\geq 0,\theta>0,\beta_2>0 \tag{4}$$

(v) The parameters λ and β_1 involved in (1) to be random variables with respective prior p.d.f.'s as

$$h_1(\lambda)=\frac{\upsilon_0^{\alpha_0}}{\Gamma\alpha_0}\, e^{-\upsilon_0\lambda}\,\lambda^{\alpha_0-1} \qquad ;(\lambda,\upsilon_0,\alpha_0)>0 \tag{5}$$

$$h_2(\beta_1)=\frac{\upsilon_1^{\alpha_1}}{\Gamma\alpha_1}\, e^{-\upsilon_1\beta_1}\,\beta_1^{\alpha_1-1} \qquad ;(\beta_1,\upsilon_1,\alpha_1)>0 \tag{6}$$

(vi) The parameters θ and β_2 involved in (2) to be random variables with respective prior
p.d.f.'s as

$$h_4(\theta)=\frac{\upsilon_2^{\alpha_2}}{\Gamma\alpha_2}\, e^{-\upsilon_2\theta}\,\theta^{\alpha_2-1} \qquad ;(\theta,\upsilon_2,\alpha_2)>0 \tag{7}$$

$$h_3(\beta_2)=\frac{\upsilon_3^{\alpha_3}}{\Gamma\alpha_3}\, e^{-\upsilon_3\beta_2}\, \beta_2^{\alpha_3-1} \qquad ;(\beta_2,\upsilon_3,\alpha_3)>0 \tag{8}$$

3. Re-modeled Stress-Strength System Reliability

The re-modeled stress-strength system reliability can be obtained as

$$R=P[U>V]$$

$$=n\beta_2\theta\int_0^\infty e^{-(m\lambda v^{\beta_1}+\theta v^{\beta_2})}\, v^{\beta_2-1}\left[1-e^{-\theta v^{\beta_2}}\right]^{(n-1)} dv \tag{9}$$

When $\beta_1=\beta_2=\beta$, then equation (9) becomes,

$$R=n\,B(n,\frac{m\lambda}{\theta}+1) \tag{10}$$

Here, it is notable that reliability function given in (10) comes out to be independent of the common shape parameter β .

4. Estimation of the Parameters and System Reliability

4.1 Classical Estimation

Let $\underset{\sim}{U}=(U_1,U_2,.....,U_{n_1})$ and $\underset{\sim}{V}=(V_1,V_2,.....,V_{n_2})$ be the random samples of sizes n_1 and n_2 respectively drawn from the distributions in (3) and (4), then the combined likelihood function of $\underset{\sim}{U}$ and $\underset{\sim}{V}$ is given by

$$\left.\begin{aligned} L(\underset{\sim}{u},\underset{\sim}{v}/\lambda,\beta_1,\beta_2,\theta)=m^{n_1}\,\beta_1^{n_1}\lambda^{n_1}\prod_{i=1}^{n_1}u_i^{\beta_1-1}\, e^{-m\lambda\sum_{i=1}^{n_1}u_i^{\beta_1}} \\ n^{n_2}\beta_2^{n_2}\,\theta^{n_2}\prod_{i=1}^{n_2}v_i^{\beta_2-1}\, e^{-\theta\sum_{i=1}^{n_2}v_i^{\beta_2}}\prod_{i=1}^{n_2}\left[1-e^{-\theta v_i^{\beta_2}}\right]^{(n-1)} \end{aligned}\right\} \tag{11}$$

The log likelihood function is

$$\left.\begin{aligned}\log L &= n_1 \log m + n_1 \log \beta_1 + n_1 \log \lambda + (\beta_1 - 1)\sum_{i=1}^{n_1} \log u_i \\ &\quad - m\lambda \sum_{i=1}^{n_1} u_i^{\beta_1} + n_2 \log n + n_2 \log \beta_2 + n_2 \log \theta \\ &\quad + (\beta_2 - 1)\sum_{i=1}^{n_2} \log v_i - \theta \sum_{i=1}^{n_2} v_i^{\beta_2} + (n-1)\sum_{i=1}^{n_2} \log[1 - e^{-\theta v_i^{\beta_2}}]\end{aligned}\right\} \tag{12}$$

The simultaneous MLE's $(\hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta})$ of $(\lambda, \beta_1, \beta_2, \theta)$ will be the solutions of the following equations:

$$\frac{\partial \log L}{\partial \lambda} = \frac{n_1}{\lambda} - m \sum_{i=1}^{n_1} u_i^{\beta_1} \tag{13}$$

$$\frac{\partial \log L}{\partial \beta_1} = \frac{n_1}{\beta_1} + \sum_{i=1}^{n_1} \log u_i - m\lambda \left(\sum_{i=1}^{n_1} u_i^{\beta_1} \log u_i\right) \tag{14}$$

$$\left.\begin{aligned}\frac{\partial \log L}{\partial \beta_2} &= \frac{n_2}{\beta_2} + \sum_{i=1}^{n_2} \log v_i - \theta\left(\sum_{i=1}^{n_2} v_i^{\beta_2} \log v_i\right) \\ &\quad + (n-1)\sum_{i=1}^{n_2}\left[\frac{1}{\left(1 - e^{-\theta v_i^{\beta_2}}\right)} e^{-\theta v_i^{\beta_2}} \; \theta v_i^{\beta_2} \; \log v_i\right]\end{aligned}\right\} \tag{15}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n_2}{\theta} - \sum_{i=1}^{n_2} v_i^{\beta_2} + (n-1)\sum_{i=1}^{n_2}\left[\frac{1}{\left(1 - e^{-\theta v_i^{\beta_2}}\right)} e^{-\theta v_i^{\beta_2}} \; v_i^{\beta_2}\right] \tag{16}$$

The equations (13), (14), (15) and (16) can be solved for $\hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2$, and $\hat{\theta}$ by using any numerical iterative procedure. Now, using invariance property of the MLE's, the MLE of the re-modeled system reliability R becomes

$$\hat{R} = n\hat{\beta}_2\hat{\theta}\int_0^\infty e^{-(m\hat{\lambda}v^{\hat{\beta}_1}+\hat{\theta}v^{\hat{\beta}_2})}\, v^{\hat{\beta}_2-1}\left[1-e^{-\hat{\theta}v^{\hat{\beta}_2}}\right]^{(n-1)} dv \tag{17}$$

Further, the asymptotic sampling distribution of $\left(\hat{\Lambda}-\Lambda\right)$ is $N_4\left(0,\Sigma^{-1}\right)$; $\Lambda = (\lambda, \beta_1, \beta_2, \theta)$. Σ is the Fisher's information matrix having elements as the expectations of the negative second partial derivatives of log-likelihood function with respect to the parameters. The asymptotic distribution of $\hat{R}$ is $N(0, A'\Sigma^{-1}A)$ where $A' = (\frac{\partial R}{\partial \lambda}, \frac{\partial R}{\partial \beta_1}, \frac{\partial R}{\partial \beta_2}, \frac{\partial R}{\partial \theta})$

.

4.2 Bayes Estimation

Here, we propose Bayesian estimation procedure by assuming the parameters of the strength and stress distributions as random variables. The prior distributions of λ, β_1, β_2 and θ are considered as $\text{Gamma}(\alpha_0, \upsilon_0)$ $\text{Gamma}(\alpha_1, \upsilon_1)$, $\text{Gamma}(\alpha_2, \upsilon_2)$ and $\text{Gamma}(\alpha_3, \upsilon_3)$ with respective p.d.fs as given in equations (5), (6), (7) and (8) respectively.

Using likelihood function in (11) and prior distributions in (5), (6), (7) and (8), the joint posterior distribution of λ, β_1, β_2, and θ given the sample is

$$\prod(\lambda, \beta_1, \beta_2, \theta \mid \underset{\sim}{u}, \underset{\sim}{v}) = L(\underset{\sim}{u}, \underset{\sim}{v} / \lambda, \beta_1, \beta_2, \theta)\, h_1(\lambda)\, h_2(\beta_1) h_3(\beta_2)\, h_4(\theta)$$

For implementing Gibbs sampling procedure, we utilize the following full conditional posterior distributions of λ, β_1, β_2 and θ :

$$\pi_1(\lambda \mid \underset{\sim}{u}, \underset{\sim}{v}, \beta_1, \beta_2, \theta) = \lambda^{n_1+\alpha_0-1}\, e^{-\lambda(\upsilon_0+m\sum_{i=1}^{n_1} u_i^{\beta_1})}$$

$$\sim G(n_1+\alpha_0, \quad \upsilon_0+m\sum_{i=1}^{n_1} u_i^{\beta_1}) \tag{18}$$

$$\pi_2(\beta_1 \mid \underset{\sim}{u}, \underset{\sim}{v}, \lambda, \beta_2, \theta) = \beta_1^{(n_1+\alpha_1-1)} \prod_{i=1}^{n_1} u_i^{\beta_1-1}\; e^{-(\upsilon_1\beta_1+m\lambda\sum_{i=1}^{n_1} u_i^{\beta_1})} \tag{19}$$

$$\left.\begin{aligned}\pi_3(\beta_2 \mid \underset{\sim}{u}, \underset{\sim}{v}, \lambda, \beta_1, \theta) = \beta_2^{n_2+\alpha_3-1} \prod_{i=1}^{n_2} v_i^{\beta_2-1}\, e^{-(\upsilon_3\beta_2+\theta\sum_{i=1}^{n_2} v_i^{\beta_2})} \\ \prod_{i=1}^{n_2}\left[1-e^{-\theta v_i^{\beta_2}}\right]^{(n-1)}\end{aligned}\right\} \quad (20)$$

$$\pi_4(\theta / , \underset{\sim}{u}, \underset{\sim}{v}, \lambda, \beta_1 \beta_2) = \theta^{(n_2+\alpha_2-1)} e^{-\theta(\upsilon_2+\sum_{i=1}^{n_2} v_i^{\beta_2})} \prod_{i=1}^{n_2}\left[1-e^{-\theta v_i^{\beta_2}}\right]^{(n-1)} \quad (21)$$

Gibbs algorithm

1. Generate λ from $\pi_1(\lambda \mid \underset{\sim}{u}, \beta_1, \beta_2, \theta)$ as given in (18).

2. Generate β_1 from the density $\pi_2(\beta_1 \mid \underset{\sim}{u}, \underset{\sim}{v}, \lambda, \beta_2, \theta)$ as given in (19).

3. Generate β_2 from the density $\pi_3(\beta_2 \mid \underset{\sim}{u}, \underset{\sim}{v}, \lambda, \beta_1, \theta)$ as given in (20).

4. Generate θ from the density $\pi_4(\theta \mid \underset{\sim}{u}, \underset{\sim}{v}, \lambda, \beta_1, \beta_2)$ as given in (21).

5. Repeat steps 1-4, M times. For eliminate the effects of the starting values, we record the sequence of $\Lambda = (\lambda, \beta_1, \beta_2, \theta)$ after N burn-in iterations i.e $(\Lambda_{N+1}, \Lambda_{N+2}, ..., \Lambda_M)$.

6. Put these generated values of Λ in the expression of R in (9).

7. The Bayes estimates Λ^* and R^* of the parameters and re-modeled system reliability and corresponding posteriors variances are taken as the means and variances of the generated values of Λ and R respectively.

8. Let $\Lambda_{(N+1)} \le \Lambda_{(N+2)} \le .... \le \Lambda_{(M)}$ and $R_{(N+1)} \le R_{(N+2)} \le .... \le R_{(M)}$ denote respectively the ordered values of $\Lambda_{N+1}, \Lambda_{N+2}, ..., \Lambda_M$, and $R_{N+1}, R_{N+2}, ..., R_M$. Then, following Chen and Shao [1998], the $100(1-\gamma)$ % HPD intervals for Λ and R respectively are $\left(\Lambda_{\left(N+j^*\right)}, \Lambda_{\left(N+j^*+\left[(1-\gamma)(M-N)\right]\right)}\right)$ and $\left(R_{\left(N+j^*\right)}, R_{\left(N+j^*+\left[(1-\gamma)(M-N)\right]\right)}\right)$ Where, i^* and j^* are chosen so that

$$\Lambda_{\left(N+j^{*}+[(1-\gamma)(M-N)]\right)}-\Lambda_{\left(N+j^{*}\right)}=\min_{N\le j\le (M-N)-[(1-\gamma)(M-N)]}\left(\Lambda_{(N+j+[(1-\gamma)(M-N)])}-\Lambda_{(N+j)}\right)$$

and

$$R_{\left(N+j^{*}+[(1-\gamma)(M-N)]\right)}-R_{\left(N+j^{*}\right)}=\min_{N\le j\le (M-N)-[(1-\gamma)(M-N)]}\left(R_{(N+j+[(1-\gamma)(M-N)])}-R_{(N+j)}\right)$$

Here, it is notable that the sampling from the posterior distributions in (19), (20) and (21) is not easy as they cannot be simplified to the well known distributions. Therefore, following Metropolis-Hastings algorithm has been used to generate β_1, β_2 and θ.

Metropolis-Hastings algorithm

1. Start with any value t_0 satisfying target density $f(t_0) > 0$.
2. Using current t value, generate a proposal point (t_prop) from the proposal density $q(t_1, t_2) = P(t_1 \to t_2)$ i.e., the probability of returning a value of t_2 given a previous value of t_1. We assume proposal densities for the distributions as $U(0, a_j)$; $j = 1, 2, 3$ respectively. The values of a_j have been set according to the corresponding assumed values of β_1, β_2 and θ.
3. Calculate the ratio at the proposal point (t_prop) and current (t_{i-1}) as:

$$\rho = \log\left[\frac{f(t\_prop)\,q(t\_prop, t_{i-1})}{f(t_{i-1})\,q(t_{i-1}, t\_prop)}\right]$$

4. Generate U from uniform on (0,1) and take Z=log U.
5. If $Z < \rho$, accept the move i.e., t_prop and set t_0=t_prop and return to step-1. Otherwise reject it and return to step-2.
6. Repeat above procedure M times.

5. A Simulation Study

For analyzing the respective values of $R_1 = P[U > V]$ in respect of m, n and involved parameters, we assume λ=0.2, $\beta_1 = 2$, $\beta_2 = 3$, $\theta = 2$. Values for $R = P[U > V]$ for varying ‘m’ and ‘n’ have been summarized in Table-1.

Table 1: Values of $R = P[U > V]$ for varying m and n

m \ n	n →			
m ↓		1	2	3	4
	1	.8950	.8582	.8368	.8221
	2	.8055	.7407	.7042	.6795
	3	.7287	.6426	.5956	.5644
	4	.6623	.5604	.5062	.4711

Here, it is observed that the re-modeled system reliability decreases as m and n increases. Now, for comparing the performances of the MLEs and Bayesian estimates of the parameters and reliability function, we generated samples of sizes n_1 and n_2 from the distributions given in (3) and (4) for the above set of values of the parameters and these samples information is used to obtain the estimates for varying combinations of m and n. Table-2 and Table-3 include the estimates of the parameters and reliability function R along with their variances/posterior variances and confidence/HPD intervals along with their widths. In Table-2 and Table-3, the entries in {}, () and [] respectively represent variances/posterior variances, confidence/HPD intervals and widths of the intervals.

6. Concluding Remarks

From table-2 and 3, it is observed that:

For fixed n and varying m:

- For all the parameters and re-modeled reliability function, Bayes estimates perform well as compared to the MLE's as they have lesser variances that of MLE's.
- Also, HPD intervals are more conservative as compared to the confidence intervals.
- Though, both the methods (ML and Bayes) are either over-estimating or under-estimating the actual re-modeled stress-strength system reliability. But preferences to be given to Bayes method of estimation as they are more consistent than MLEs.

For fixed m and varying n:

- For varying n, trends in estimating the parameters and re-modeled reliability function are exactly the same as observed in case of varying m.

7. Real Data Analysis

In this section, we present a data analysis of the strength data reported by Badar and Priest (1982). The data represent the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purpose, we consider the data sets consisting the single fibers of 20 mm (Data Set 1) and 10 mm (Data Set 2 belongs to stress measurements) in gauge lengths with sample sizes 67 and 63 respectively. Data sets are presented below:

Data set 1: (belongs to strength measurements**)**

U= [.312, .314, .479, .552, .7, .803, .861, .865, .944, .958, .966, .997, 1.006, 1.021, 1.055, 1.063, 1.098, 1.14, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633,1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954,2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585]

Data set 2: (belongs to stress measurements**)**

V=[.101, .332, .403, .428, .457, .550, .561, .596, .597, .645, .654, .674, .718, .722, .725, .732,.775, .814, .816, .818, .824, .859, .875, .938, .940,1.056,1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230 ,1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546,1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2.052, 2.071, 2.086, 2.171,2.224,2.227, 2.425, 2.595, 3.22]

Using the above data sets, Firstly, we obtain the MLEs of the parameters and re-modeled system reliability R for varying values of m and n. The values of the MLEs along with their variances and 95% confidence intervals are reported in Table-4. In Table-4, the values in the brackets {}, () and [] respectively represent the variances, confidence intervals and P-value of the corresponding K-S distance. K-S1 and K-S2 stand for Kolmogorov-Smirnov distances for data set 1 and 2 respectively. Secondly, for testing the goodness-of-fit of the data sets for the proposed model, we consider the following three methods:

- Kolmogorov-Smirnov (K-S) distance between the fitted and the empirical distribution functions.
- The P-P plots of fitted and empirical distribution functions.
- The plots of fitted verses empirical distribution functions.

Kolmogorov-Smirnov (K-S) distances between the fitted and the empirical distribution functions for data sets 1 and 2 for varying m and n with corresponding P-values are summarized in Table-4. The P-P plots of fitted and empirical distribution functions and the plots of fitted verses empirical distribution functions are also drawn in Fig.-1 to Fig.-16. In all these figures, the left pairs of the plots are of data set-1 whereas the right pairs of the plots belong to the data set-2.

The P-values of the K-S differences show that both the considered data sets fit well to the proposed model for all considered combinations of the measurements m and n. The P-P plots and plots of fitted verses empirical distribution functions have also provided the same evidences.

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Table 2: MLE's and Bayes estimates along with their variances / posterior variances and confidence / HPD intervals for fixed n and varying m

Actual	m=1, n=4		m=2, n=4	
	MLE's	Bayes	MLE's	Bayes
$\lambda=0.2$	.1802{.0023} (.0859,.2746) [.1887]	.1913{.0016} (.1137,.2623) [.1486]	.1952{.0016} (.1154,.2751) [.1597]	.1978{.0012} (.1307,.2656) [.1349]
$\beta_1=2$	2.0622{.0465} (1.6391, 2.4852) [.8461]	2.0314{.0287} (1.7197, 2.3054) [.5857]	2.0433{.0553} (1.5821, 2.5045) [.9924]	2.0321{.0357} (1.6122, 2.3644) [.7522]
$\beta_2=3$	3.2424{.1145} (2.5791, 3.9057) [1.3266]	3.1128{.0635} (2.6518, 3.6523) [1.0005]	3.4367{.1267} (2.7391, 4.1344) [1.3953]	3.2166{.0672} (2.7636, 3.7169) [.9533]
$\theta=2$	1.8582{.0232} (1.5594, 2.1570) [.5976]	1.9632{.0191} (1.6956, 2.2192) [.5236]	1.9258{.0245} (1.6190, 2.2326) [.6136]	1.9748{.0194} (1.7167, 2.2405) [.5238]
Actual R	.8221	.6795		
Estimated R	.8312{.0055} (.6849,.9774) [..[.2925]	.8279{.0011} (.7678..8902) [.1224]	.6799{.0053} (.5369,.8229) [.2860]	.6808{.0021} (.5925,.7709) [.1784]

Actual	m=3, n=4		m=4, n=4	
	MLE's	Bayes	MLE's	Bayes
$\lambda=0.2$	.1640{.0009} (.1032,.2250) [.1218]	.1841{.0008} (.1250,.2364) [.1114]	.1937{.0009} (.1328,.2546) [.1218]	.1950{.0008} (.1469,.2593) [.1124]
$\beta_1=2$	2.2381{.0834} (2.0463, 3.1784) [1.1321]	2.1057{.0452} (1.9331, 2.7858) [.8527]	2.0326{.0498} (1.5952, 2.4701) [.8749]	2.0127{.0319} (1.6952, 2.4080) [.7128]
$\beta_2=3$	3.3020{.1157} (2.6352, 3.9688) [1.3356]	3.1380{.0587} (2.6401, 3.5742) [.9341]	3.1096{.1005} (2.4881, 3.7311) [1.2430]	3.0656{.0603} (2.6300, 3.5371) [.9071]
$\theta=2$	1.7982{.0220} (1.5069, 2.0895) [.5826]	1.9634{.0213} (1.6888, 2.2404) [.5516]	2.0022{.0260} (1.6857, 2.3188) [.6331]	2.0022{.0186} (1.7495, 2.2453) [.4958]
Actual R	.5644	.4711		
Estimated R	.5991{.0044} (.4745,.7413) [.2668]	.5882{.0024} (.4947,.6824) [.1877]	.4817{.0037} (.3614,.6021) [.2407]	.4784{.0027} (.3748,.5746) [.1998]

Table 3: MLE's and Bayes estimates along with their variances / posterior variances and confidence / HPD intervals for fixed m and varying n

Actual	m=4, n=1		m=4, n=2	
	MLE's	Bayes	MLE's	Bayes
$\lambda=0.2$	.2167{.0011} (.1556,.2824) [.1268]	.2138{.0009} (.1556,.2762) [.1206]	.2278{.0012} (.1596,.2960) [.1364]	.2302{.0010} (.1737,.3002) [.1265]
$\beta_1=2$	1.9152{.0477} (1.4870,2.3434) [.8564]	1.9393{.0326} (1.6227,2.3301) [.7074]	2.3292{.0651} (1.8288,2.8297) [1.0009]	2.2178{.0419} (1.8663,2.6011) [.7348]
$\beta_2=3$	2.9371{.1025} (1.4974,2.6872) [1.1898]	2.9395{.0599} (2.4809,3.4343) [.9534]	3.0992{.1091} (2.4517,3.7467) [1.2950]	3.0313{.0613} (2.5055,3.4911) [.9856]
$\theta=2$	2.0932{.0921} (1.4974,2.6872) [1.1898]	1.3662{.0229} (1.0991,1.4956) [..3965]	1.9997{.0447} (1.5852,2.4142) [.8290]	1.5775{.0188} (1.3310,1.8138) [.4828]
Actual R	0.6623	.5604		
Estimated R	.6466{.2602} (0.5635,.8756) [.3121]	.5690{.0023} (.4952,.6736) [.1784]	.5317{.0283} (.2011,.8616) [.6605]	.4734{.0026} (.3764,.5654) [.1890]

Actual	m=4, n=3		m=4, n=4	
	MLE's	Bayes	MLE's	Bayes
$\lambda=0.2$	.2545{.0014} (.1803,.3286) [.1483]	.2480{.0011} (.1776,.3088) [.1312]	.1937{.0009} (.1328,.2546) [.1218]	.1950{.0008} (.1469,.2593) [.1124]
$\beta_1=2$	1.9605{.0471} (1.5350,2.3860) [.8510]	1.9704{.0310} (1.6421,2.3128) [.6707]	2.0326{.0498} (1.5952,2.4701) [.8749]	2.0127{.0319} (1.6952,2.4080) [.7128]
$\beta_2=3$	3.0196{.1058} (2.4540,3.7291) [1.2751]	2.9616{0625} (2.4658,3.4618) [.9960]	3.1096{.1005} 2.4881,3.7311) [1.2430]	3.0656{.0603} (2.6300,3.5371) [.9071]
$\theta=2$	2.3133{.0442} (1.9012,2.7255) [.8243]	1.8031{.0190} (1.5137,2.0246) [.5109]	2.0022{.0260} (1.6857,2.3188) [.6331]	2.0022{.0186} (1.7495,2.2453) [.4958]
Actual R	.5062	.4711		
Estimated R	.4547{.0107} (.2513,.6582) [.4069]	.4133{.0025} (.3310,.5210) [.1900]	.4817{.0037} (.3614,.6021) [.2407]	.4784{.0027} (.3748,.5746) [.1998]

Table 4: MLE's of the parameters and reliability function along with their variances and confidence intervals, K-S test's values with corresponding P-values for varying m and n (For data set 1 & 2)

n / m	N →				
	MLE's	1	2	3	4
m	$\hat{\lambda}$	0.2057{.0021} (.1140,.2973)	0.2057{.0021} (.1140,.2973)	0.2057{.0021} (.1140,.2973)	0.2057{.0021} (.1140,.2973)
	$\hat{\beta}_1$	3.2499{.0979} (2.6366,3.8632)	3.2499{.0979} (2.6366,3.8632)	3.2499{.0979} (2.6366,3.8632)	3.2499{.0979} 2.6366,3.8632)
1 ↓	$\hat{\beta}_2$	2.1556{.0439} (1.7446,2.5666)	1.4930{.0203} (1.2136,1.7724)	1.2257{.0134} (.9985,1.4529)	1.0741{.0101} (.8762,1.2720)
	$\hat{\theta}$	0.4677{.0046} (.3107,.6247)	0.9761{.0115} (.7650,1.1871)	1.3325{.0139} (1.1012,1.5638)	1.6022{.0152} 1.3604,1.8440)
	$\hat{R}_1$	0.6126	0.6229	0.6273	0.6297
	K-S1[p-value]	0.0524[.9921]	0.0524[.9921]	0.0524[.9921]	0.0524[.9921]
	K-S2[p-value]	0.0693[.9230]	0.0806[.8075]	0.0894[.6957]	0.0940[.6334]
	$\hat{\lambda}$	0.1028{.0005} (.0569,.1487)	0.1028{.0005} (.0569,.1487)	0.1028{.0005} (.0569,.1487)	0.1028{.0005} (.0569,.1487)
	$\hat{\beta}_1$	3.2499{.0981} (2.6359,3.8640)	3.2499{.0981} (2.6359,3.8640)	3.2499{.0981} 2.6359,3.8640)	3.2499{.0981} 2.6359,3.8640)
2	$\hat{\beta}_2$	2.1556{.0439} (1.7446,2.5666)	1.4930{.0203} (1.2136,1.7724)	1.2257{.0134} (.9985,1.4529)	1.0741{.0101} (.8762,1.2720)
	$\hat{\theta}$	.4677{.0046} (.3107,.6247)	0.9761{.0115} (.7650,1.1871)	1.3325{.0139} 1.1012,1.5638)	1.6022{.0152} 1.3604,1.8440)
	$\hat{R}_1$	0.6126	0.6229	0.6273	0.6296
	K-S1[p-value]	0.0523[.9924]	0.0523[.9924]	0.0523[.9924]	0.0523[.9924]
	K-S2[p-value]	0.0693[.9230]	0.0806[.8075]	0.0894[.6957]	0.0940[.6334]
	$\hat{\lambda}$	0.0685{.0002} (.0379,.0992)	0.0685{.0002} (.0379,.0992)	0.0685{.0002} (.0379,.0992)	0.0685{.0002} (.0379,.0992)
	$\hat{\beta}_1$	3.2499{.0983} (2.6351,3.8647)	3.2499{.0983} (2.6351,3.8647)	3.2499{.0983} 2.6351,3.8647)	3.2499{.0983} 2.6351,3.8647)
3	$\hat{\beta}_2$	2.1556{.0439} (1.7446,2.5666)	1.4930{.0203} (1.2136,1.7724	1.2257{.0134} (.9985,1.4529)	1.0741{.0101} (.8762,1.2720)
	$\hat{\theta}$	0.4677{.0046} (.3107,.6247)	0.9761{.0115} (.7650,1.1871)	1.3325{.0139} 1.1012,1.5638)	1.6022{.0152} 1.3604,1.8440)
	$\hat{R}_1$	0.6126	0.6229	0.6273	0.6296
	K-S1[p-value]	0.0521[.9927]	0.0521[.9927]	0.0521[.9927]	0.0521[.9927]
	K-S2[p-value]	0.0693[.9230]	0.0806[.8075]	0.0894[.6957]	0.0940[.6334]
	$\hat{\lambda}$	0.0514{.0001} (.0284,.0744)	0.0514{.0001} (.0284,.0744)	0.0514{.0001} (.0284,.0744)	0.0514{.0001} (.0284,.0744)
	$\hat{\beta}_1$	3.2499{.0986} (2.6344,3.8655)	3.2499{.0986} (2.6344,3.8655)	3.2499{.0986} 2.6344,3.8655)	3.2499{.0986} 2.6344,3.8655)
4	$\hat{\beta}_2$	2.1556{.0439} (1.7446,2.5666)	1.4930{.0203} (1.2136,1.7724	1.2257{.0134} (.9985,1.4529)	1.0741{.0101} (.8762,1.2720)
	$\hat{\theta}$	0.4677{.0046} (.3107,.6247)	0.9761{.0115} (.7650,1.1871)	1.3325{.0139} 1.1012,1.5638)	1.6022{.0152} 1.3604,1.8440)
	$\hat{R}_1$	0.6126	0.6229	0.6273	0.6296
	K-S1[p-value]	0.0523[.9924]	0.0523[.9924]	0.0523[.9924]	0.0523[.9924]
	K-S2[p-value]	0.0693[.9230]	0.0806[.8075]	0.0894[.6957]	0.0940[.6334]

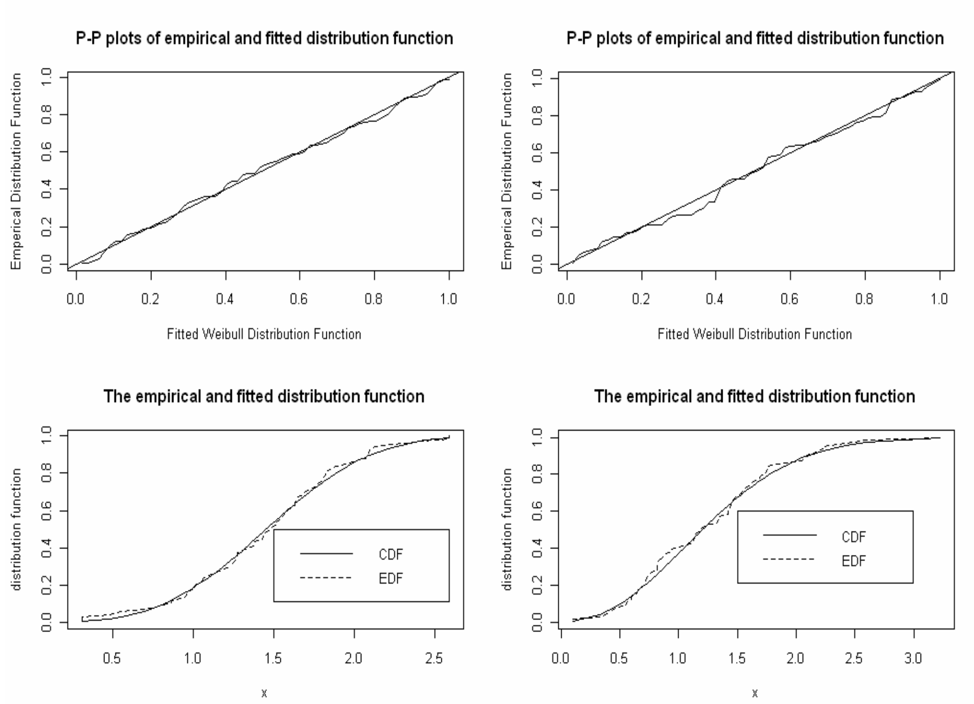

Fig-1: Plots for m=1 and n=1

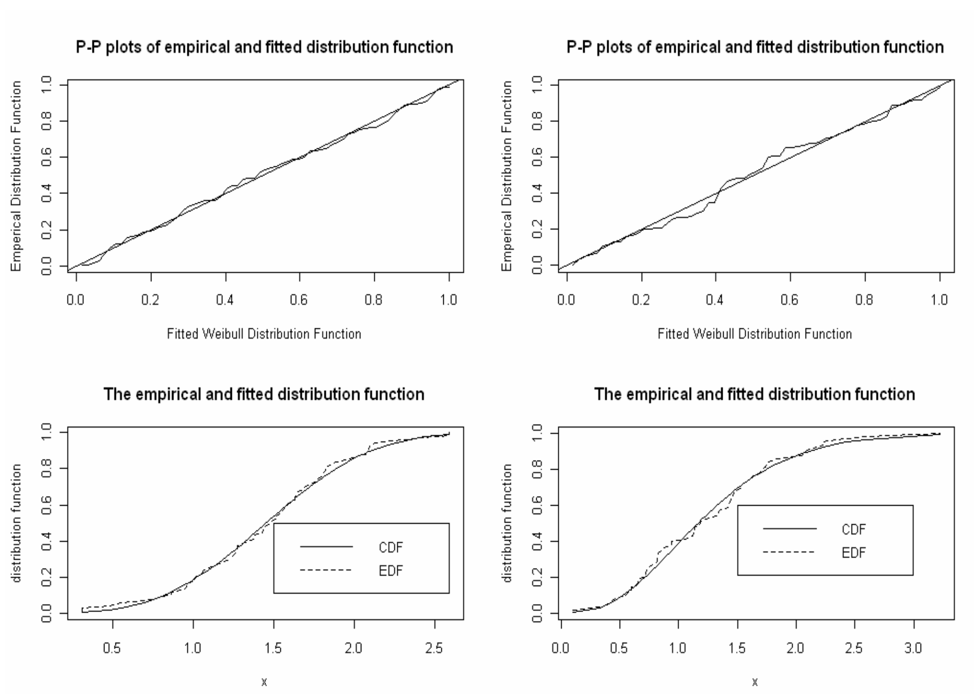

Fig-2: Plots for m=1 and n=2

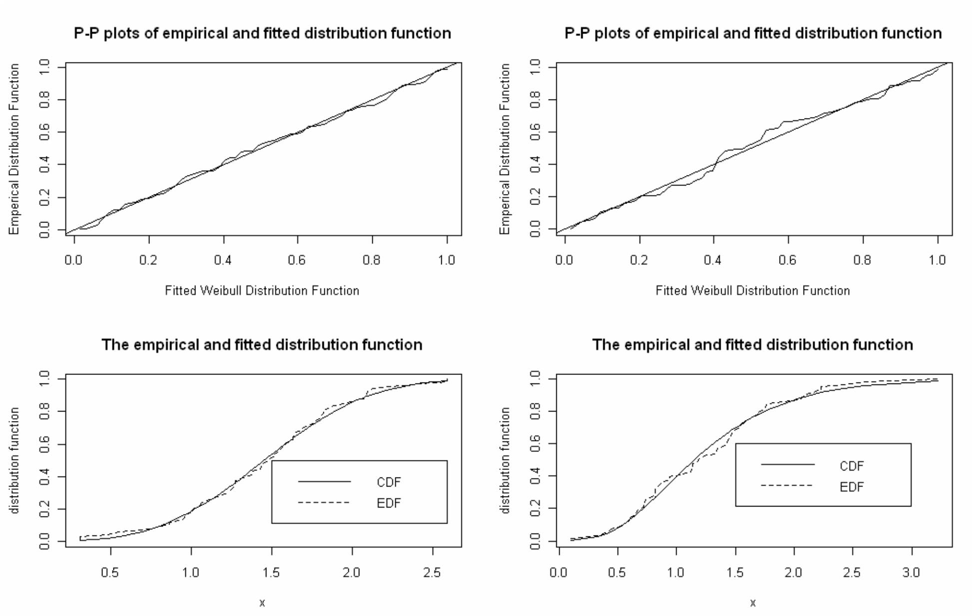

Fig-3: Plots for m=1 and n=3

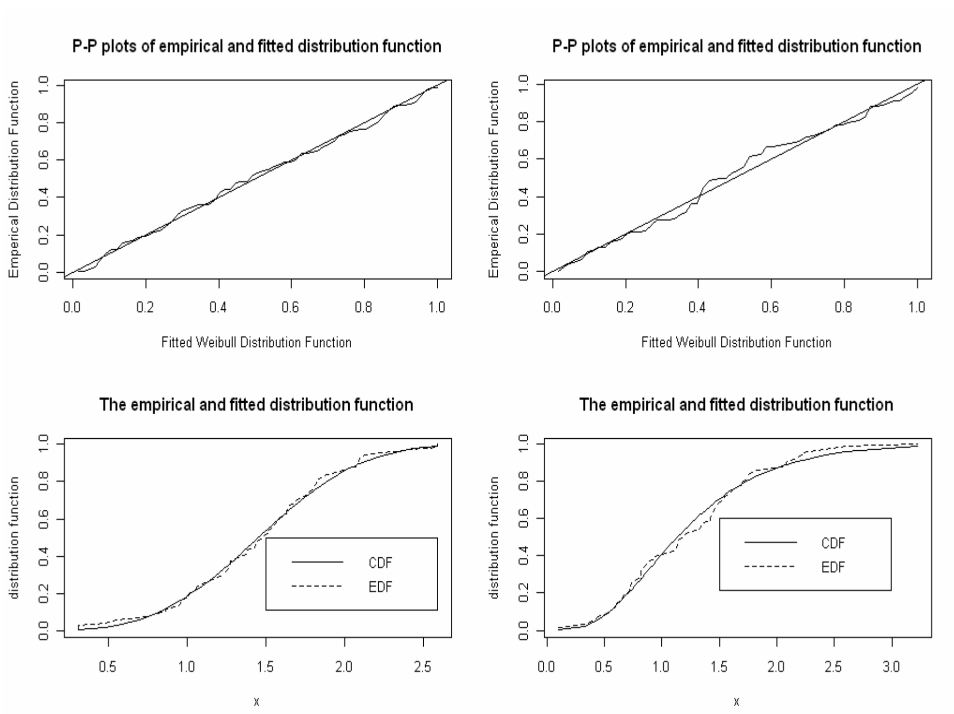

Fig-4: Plots for m=1 and n=4

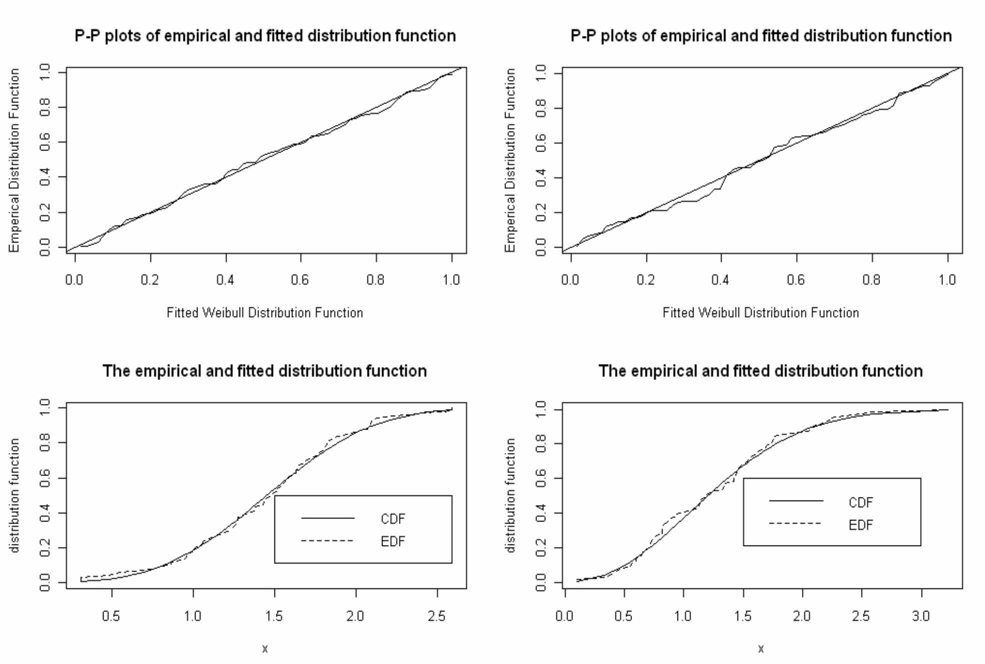

Fig-5: Plots for m=2 and n=1

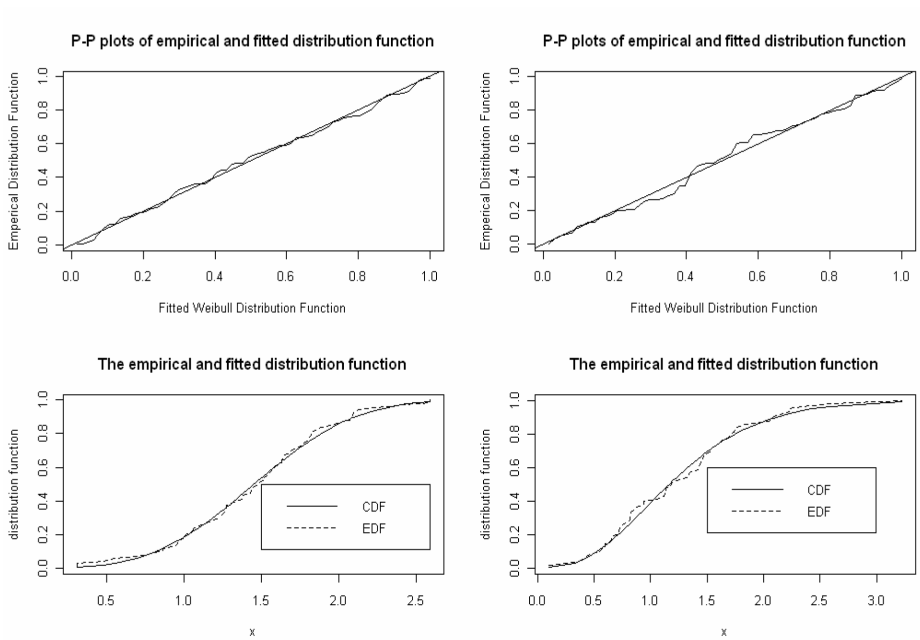

Fig-6: Plots for m=2 and n=2

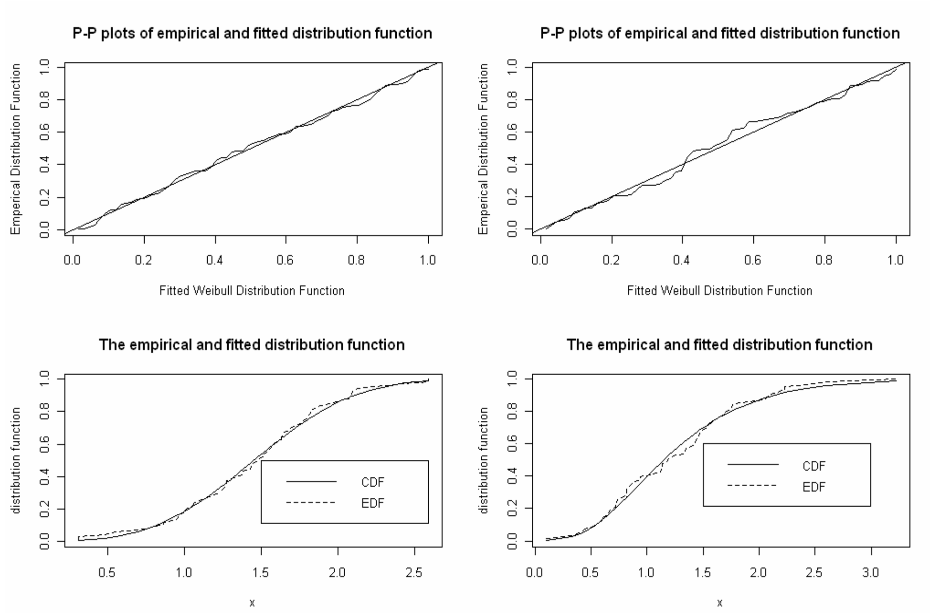

Fig-7: Plots for m=2 and n=3

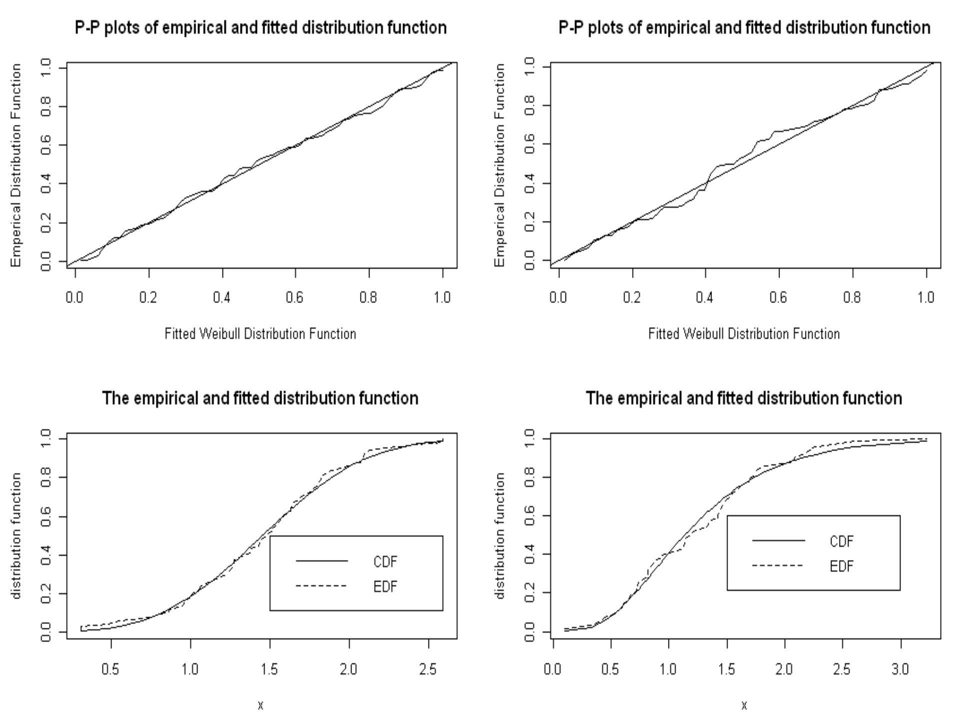

Fig-8: Plots for m=2 and n=4

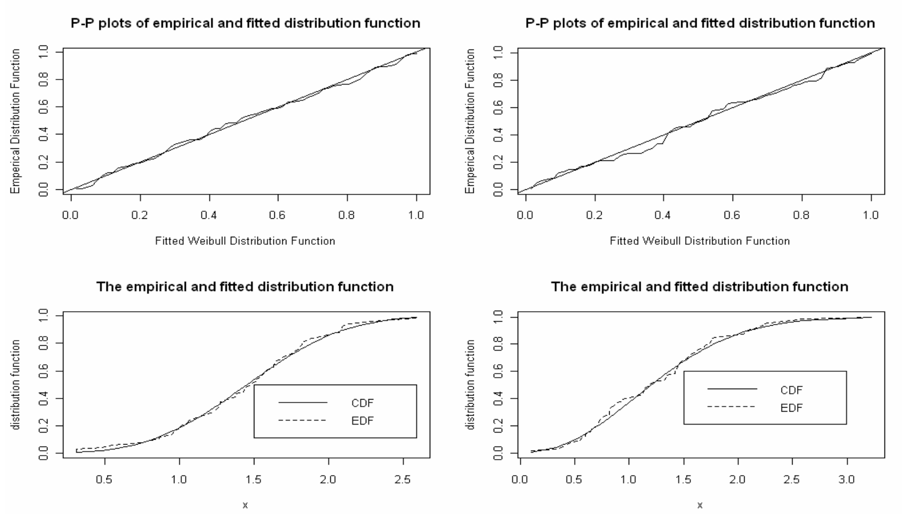

Fig-9: Plots for m=3 and n=1

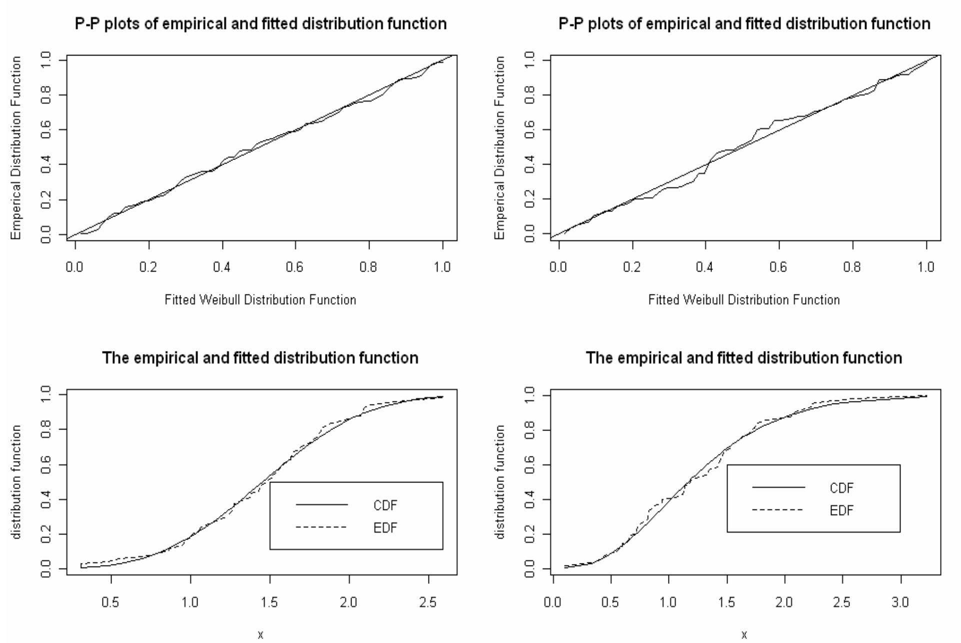

Fig-10: Plots for m=3 and n=2

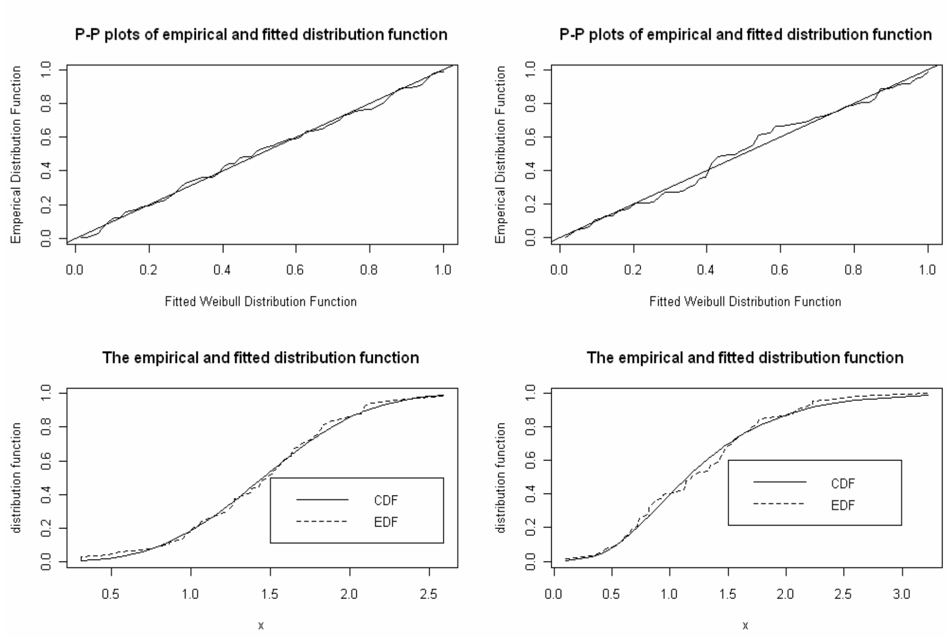

Fig-11: Plots for m=3 and n=3

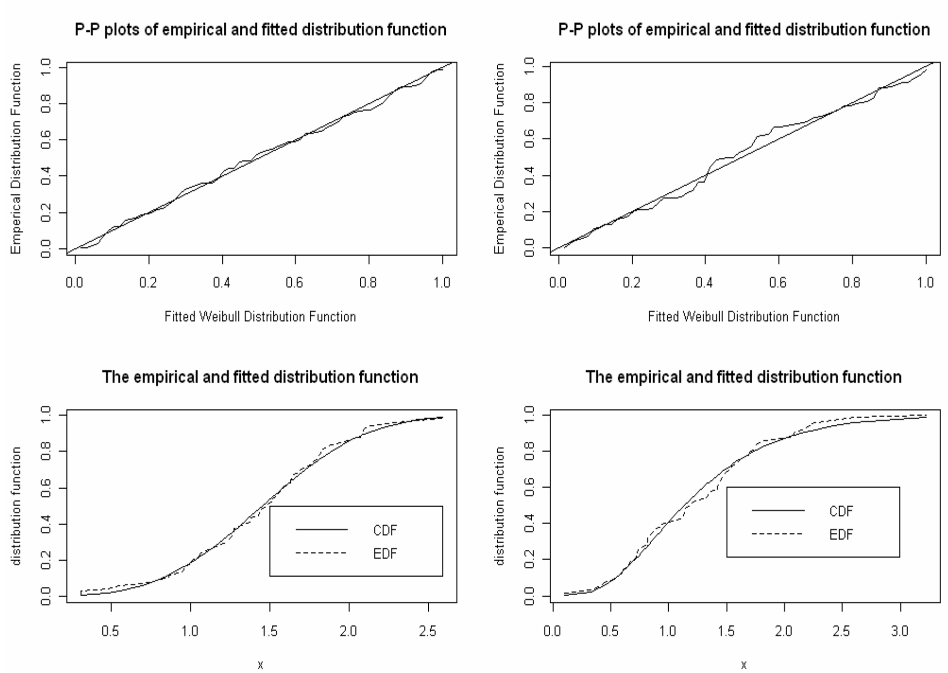

Fig-12: Plots for m=3 and n=4

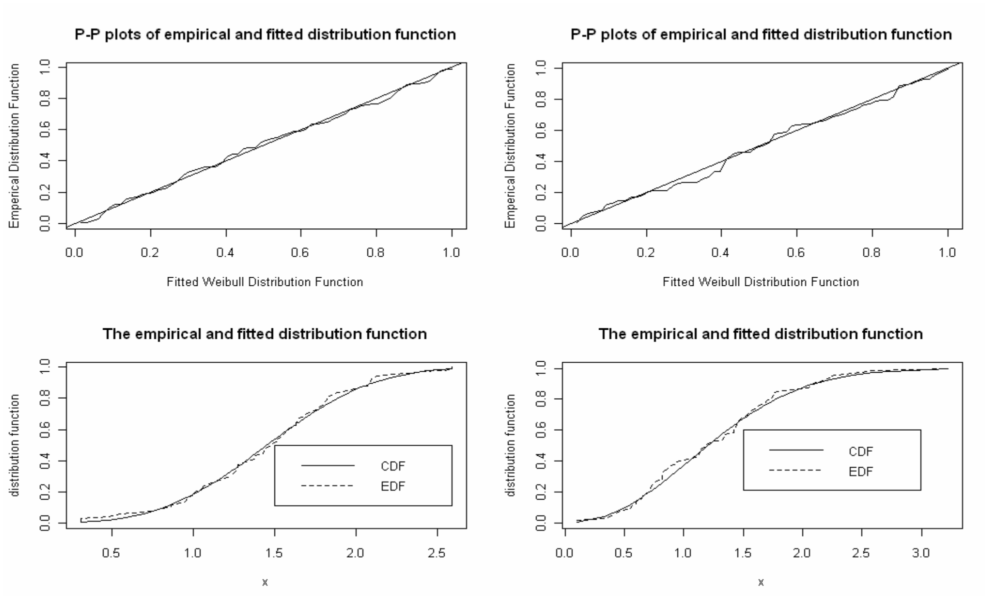

Fig-13: Plots for m=4 and n=1

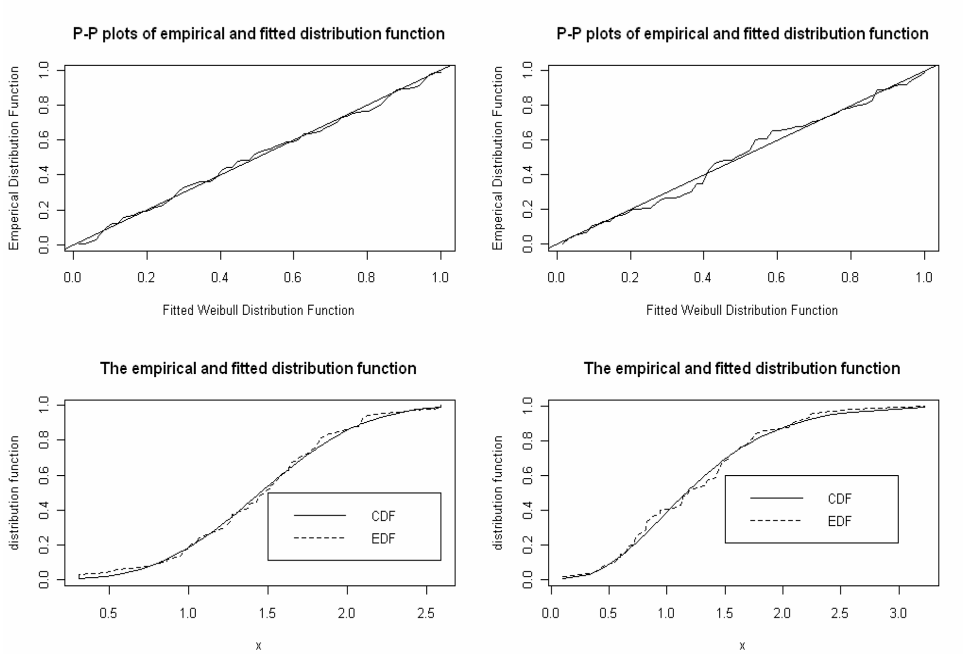

Fig-14: Plots for m=4 and n=2

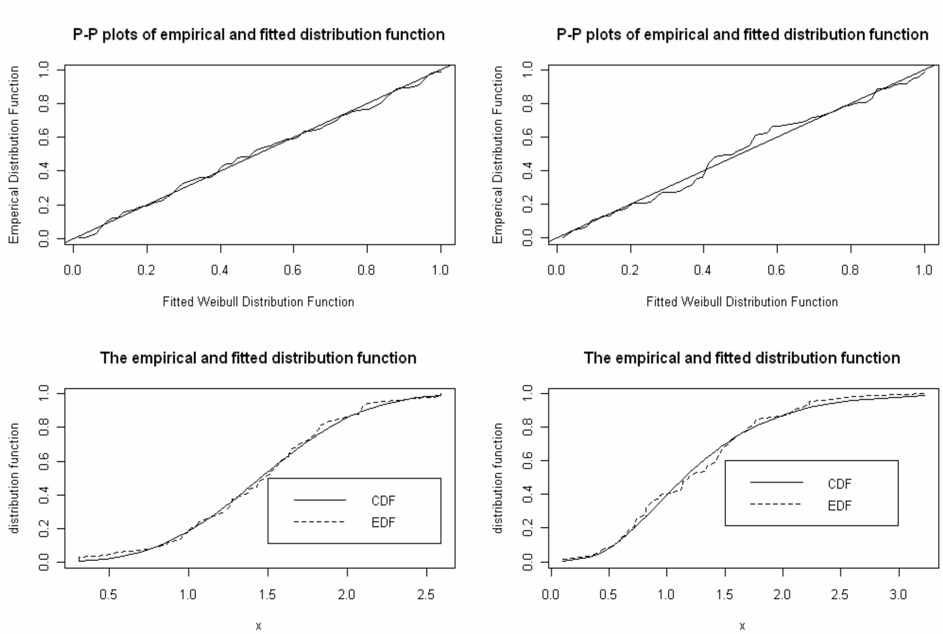

Fig-15: Plots for m=4 and n=3

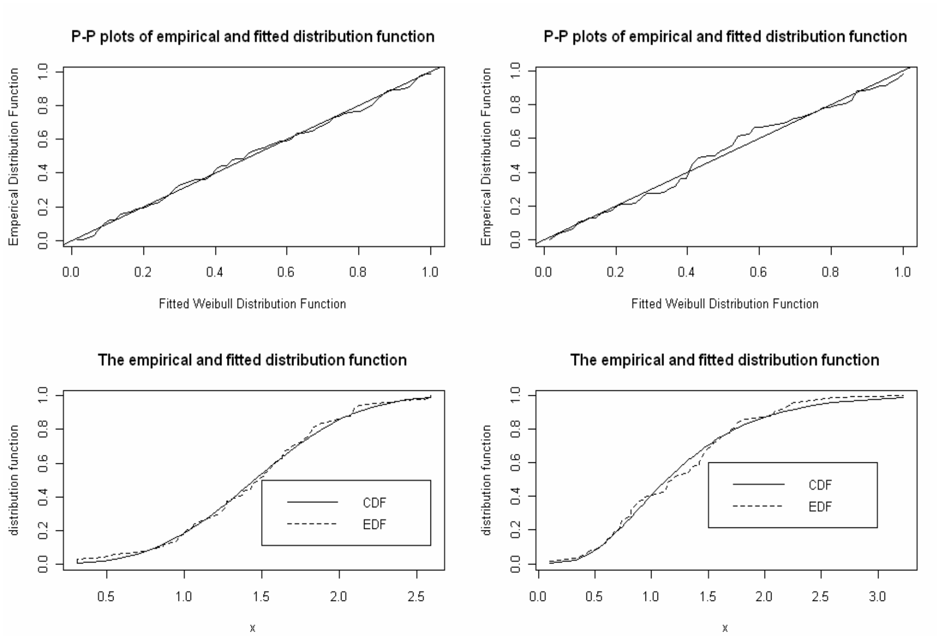

Fig-16: Plots for m=4 and n=4