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MOMENTS AND JOINT DISTRIBUTION OF CONCOMITANTS OF ORDER STATISTICS

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Abstract

Present article deals in the mathematical development of moments and joint distribution of concomitants of order statistics. Gumbel's bi-variate exponential distribution (1960) has been considered as the parental distribution of the order statistics. Idea behind this article is to contribute in developing mathematical expressions which can directly be used as a tool.

Key words: Bivariate distribution, order statistics, concomitant,

1. Introduction

Concomitant is a Latin word which stands for 'accompanying or associated'. Concomitants can be defined in brief as follows;
Let (X_r,Y_r); r=1,2,....., n be a random sample of n pairs drawn from a bivariate normal $N(\theta_x,\theta_y,\sigma_x^2,\sigma_y^2,\rho)$ distribution. When the X's are arranged in ascending order as;

$$X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq \dots\dots$$

We denote the corresponding paired Y's by

$$Y_{[1:n]}, Y_{[2:n]}, Y_{[3:n]}, \dots\dots$$

These are not arranged in any order but are the paired values of the corresponding order statistics. That is $Y_{[2,i]}$ is the value from second sample corresponds to i^{th} order statistics. Thisis called concomitant of ith order statistics.

In mathematical or statistical studies the use of concomitants arises in selection procedures when few of the individuals are chosen on the basis of their one characteristic and the corresponding (or induced or associated) characteristic to those individuals is forced to be considered. Or if we select individuals having the highest X-scores, we may wish to to know the behavior of the corresponding Y-scores. The second characteristic which comes in selection as an induced effect has reasonable effect on the study. Study of concomitants helps in measuring the probabilistic behavior of such situation. Keeping the above utilization of concomitants in mind the present article has been written. Following are the distribution and probability density functions due to Gumbel (1960);

If (x_i,y_i), i=1,2,....,n, be n pairs of independent random variables from Gumbel's exponential bivariate distribution (which may be considered as a particular case of Gumbel's weibull bivariate distribution) with the distribution function [Johnson and Kotz (1972)]

$$F(x,y) = 1 - e^{-ax} - e^{by} + e^{(-ax-by-cxy)}$$

$$x,y \geq 0, 0 \leq c \leq ab \text{ and } a,b, > 0 \tag{1}$$

and the probability density function ;

$$f(x,y) = [(b+cx)(a+cy) - c]e^{(-ax-by-cxy)}$$

$$x,y \geq 0, 0 \leq c \leq ab \text{ and } a,b, > 0 \tag{2}$$

2. Marginal Probability Functions

The marginal probability density function of X comes out as;

$$g(x) = ae^{-ax}; \quad a,x > 0 \tag{3}$$

And the marginal distribution function of X is;

$$G(x) = 1 - e^{-ax}, \quad a,x > 0 \tag{4}$$

Similarly, the marginal probability density function of Y is;

$$h(y) = be^{-by}; \; b,y > 0 \tag{5}$$

And the marginal distribution function of Y is;

$$H(y) = 1 - e^{-by}; \; b,y > 0 \tag{6}$$

With;

$$E(X) = \frac{1}{a} \tag{7}$$

$$E(X^2) = \frac{2}{a^2} \tag{8}$$

$$E(Y) = \frac{1}{b} \tag{9}$$

$$E(Y^2) = \frac{2}{b^2} \tag{10}$$

3. Conditional Probability Density Functions

The conditional probability density function of Y for given X can be obtained as follows;

$$f(y/x)=\left\{\frac{[(b+cx)(a+cy)-c]\,e^{-y(b+cx)}]}{a}\right\} \tag{11}$$

Similarly, the conditional probability density function of X for given Y will as follows;

$$f(x/y)=\left\{\frac{[(b+cx)(a+cy)-c]\;e^{-x(a+cy)}\;]}{b}\right\} \tag{12}$$

4. Probability Density Function of Order Statistics

For Gumbel's exponential bivariate distribution the probability density function of the r[th] order statistics $X_{r:n}$ is;

$$f_{r:n}(x)=C_{r:n}[1-e^{-ax}]^{r-1}\,a\,e^{-a(n-r+1)x} \tag{13}$$

where $C_{r:n}=\dfrac{n!}{(r-1)!\;(n-r)!}$

In particular for r=1, i.e. the probability density function of the first order statistics is

$$f_{1:n}(x)=n\,a\;e^{-nax} \tag{14}$$

For the distribution with probability density function (3), the joint distribution of r[th] and s[th] order statistics comes out as follows;

$$f_{r,s:n}(x_1,x_2)=C_{r,s:n}a^2\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{(i+j)}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$e^{-(ia+sa-ra-ja)x_1}\,e^{-(ia+sa+ja+a)x_2} \tag{15}$$

where $C_{r,s:n}=\dfrac{n!}{(r-1)!\;(s-r-1)!\;(n-s)!}$

5. Probability Density Function of Concomitants

Now, the probability density function of the first order concomitant (i.e. r=1) of the order statistics is; [David (1981)]

$$g_{[1\,:\,n]}(y)=\int_0^{\infty}f(y/x)\,f_{1:n}(x)dx$$

$$= n\, e^{-by} \int_0^{\infty} (cx(a+cy)+ab-c+bcy)\, e^{-x(na+cy)} dx$$

Now using the following relation,

$$\int_0^{\infty} \exp(-pt)t^{v} dt = p^{-(v+1)}\Gamma(v+1) \qquad \text{Re } v \geq -1, \quad \text{Re } p \geq 0$$

And after simple steps we get,

$$g_{[1:n]}(y) = e^{(-by)\left\{nb - \frac{abn(n-1)}{c\left(y+\frac{na}{c}\right)} - \frac{an(n-1)}{c\left(y+\frac{na}{c}\right)^2}\right\}} ; \quad y \geq 0 \tag{16}$$

Similarly, for x, the pdf $g_{[1:n]}(x)$ is obtained as;

$$g_{[1:n]}(x) = e^{(-ax)\left\{na - \frac{abn(n-1)}{c\left(x+\frac{nb}{c}\right)} - \frac{an(n-1)}{c\left(x+\frac{nb}{c}\right)^2}\right\}} , \quad x \geq 0 \quad . \tag{17}$$

Now, the probability density function of the rth concomitant can be obtained as;

$$g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r}\binom{n}{i}\; g_{[1:i]}(y)$$

$$g_{[r\,:\,n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r}\binom{n}{i}\; e^{(-by)\left\{ib - \frac{abi(i-1)}{c\left(y+\frac{ia}{c}\right)} - \frac{ai(i-1)}{c\left(y+\frac{ia}{c}\right)^2}\right\}} \tag{18}$$

Similarly, we can obtain,

$$g_{[r:n]}(x) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r}\binom{n}{i}\; e^{(-ax)\left\{ia - \frac{abi(i-1)}{c\left(x+\frac{ib}{c}\right)} - \frac{bi(i-1)}{c\left(x+\frac{ib}{c}\right)^2}\right\}} \tag{19}$$

6. Moments of Concomitants

The k^{th} moment about origin of the first concomitant i.e. of $Y_{[1:n]}$ is given by,

$$\mu^k_{y[1:n]} = \int_0^\infty y^k g_{[1:n]}(y)\ dy$$

$$= n\frac{\Gamma(k+1)}{b^k} - \frac{abn(n-1)}{c}\left\{\int_0^\infty \exp(-by)y^k\left(y+\frac{na}{c}\right)^{-1} dy - \frac{1}{b}\int_0^\infty \exp(-by)y^k\left(y+\frac{na}{c}\right)^{-2} dy\right\}$$

Integrating by parts and using the following relation,

$$\int_0^\infty e^{(-pt)t^\nu}(t+\alpha)^{-1}dt = \Gamma(\nu+1)\alpha^\nu e^{\alpha,p}\Gamma(-\nu,\alpha\ p)$$

$$\{|\alpha| < \pi, \nu > 1, p > 0\}$$

we get,

$$\mu^k_{y[1:n]} = n\frac{\Gamma(k+1)}{b^k} - \left(\frac{na}{c}\right)^k (n-1)\Gamma(k+1)e^{\left(nab/c\right)}\Gamma\left(1-k, nab/c\right) \tag{20}$$

Now, the k^{th} moment about origin of $Y_{[r:n]}$ will be;

$$\mu^k_{y[r:n]} = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\ \mu^k_{y[1:\mathrm{i}]}$$

$$\mu^k_{y[r:n]} = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}$$

$$\left\{i\frac{\Gamma(k+1)}{b^k} - \left(\frac{ia}{c}\right)^k (i-1)\Gamma(k+1)e^{\left(iab/c\right)}\Gamma\left(1-k, iab/c\right)\right\} \tag{21}$$

Similarly, the k^{th} moment about origin of $X_{[r:n]}$ can be obtained as

$$\mu^k_{x[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ i\frac{\Gamma(k+1)}{a^k} - \left(\frac{ib}{c}\right)^k (i-1)\Gamma(k+1) e^{\left(iab/c\right)} \Gamma\left(1-k, iab/c\right) \right\} \qquad (22)$$

Now, in particular for k=1 the mean of $Y_{[r:n]}$ will be;

$$E(Y_{[r:n]}) = \mu^1_{y[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ \frac{i}{b} - \left(\frac{ia}{c}\right) (i-1) e^{\left(iab/c\right)} \Gamma\left(0, iab/c\right) \right\} \qquad (23)$$

The variance of $Y_{[r:n]}$ can be obtained as

$$V(Y_{[r:n]}) = \mu^2_{y[r:n]} - (\mu^1_{y[r:n]})^2$$

where

$$\mu^2_{y[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ \frac{2}{b^2} - 2\left(\frac{ia}{c}\right)^2 (i-1) e^{\left(iab/c\right)} \Gamma\left(-1, iab/c\right) \right\} \qquad (24)$$

Similarly, the mean of $X_{[r:n]}$ will be;

$$E(X_{[r:n]}) = \mu^1_{x[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i}$$

$$\left\{ \frac{i}{a} - \left(\frac{ib}{c}\right) (i-1) e^{\left(iab/c\right)} \Gamma\left(0, iab/c\right) \right\} \qquad (25)$$

The variance of $X_{[r:n]}$ can be obtained as

$$V(X_{[r:n]}) = \mu^2_{x[r:n]} - (\mu^1_{x[r:n]})^2$$

where

$$\mu^2_{x[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r}\binom{n}{i} \left\{ \frac{2}{a^2} - 2\left(\frac{ib}{c}\right)^2 (i-1) e^{\left(iab/c\right)} \Gamma\left(-1, iab/c\right) \right\} \quad (26)$$

7. Joint Distribution of Two Concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$

Now, for the Gumbel's bivariate exponential distribution with probability density function (2), the joint distribution of two concomitants of order statistics $Y_{[r:n]}$ and $Y_{[s:n]}$, which is denoted as $g_{[r,s,n]}(y_1, y_2)$, can be obtained as follow;

$$f_{Y[r:n],Y[s:n]}(y_1, y_2) = g_{[r,s:n]}(y_1, y_2)$$

$$g_{[r,s:n]}(y_1, y_2) = \int_0^\infty \int_0^{x_2} f\left(\frac{y_1}{x_1}\right) f\left(\frac{y_2}{x_2}\right) f_{r,s:n}(x_1, x_2) dx_1 dx_2$$

where $f_{r,s:n}(x_1, x_2)$ is the joint probability density function of r[th] and s[th] order statistics of X.

$$= \int_0^\infty \int_0^x 2 \frac{[(b+cx_1)(a+cy_1)-c]\, e^{[-y_1(b+cx_1)]}}{a} \frac{[(b+cx_2)(a+cy_2)-c]\, e^{[-y_2(b+cx_2)]}}{a}$$

$$C_{r,s:n}\; a^2 \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{(i+j)} \binom{r-1}{i}\binom{s-r-1}{j} e^{[-(ia+sa-ra-ja)x_1]}$$

$$e^{[-(na-sa+ja+a)x_2]} dx_1 dx_2$$

$$= C_{r,s:n}\; e^{[-b(y_1+y_2)]} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{(i+j)} \binom{r-1}{i}\binom{s-r-1}{j}$$

$$\left\{ \int_0^\infty [(b+cx_2)(a+cy_2) - c]\, e^{[-(cy_2+na-sa+ja+a)x_2]} I(x_2, y_1) dx_2 \right\} \quad (27)$$

where,

$$I(x_2, y_1) = \int_0^{x_2} [(b+cx_1)(a+cy_1) - c]\, e^{-(cy_1+ia+sa-ra-ja)x_1} dx_1$$

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