Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online):2229-5666
Vol. 4, Issue 1 (2011): 53-63

IMPROVED RATIO TYPE ESTIMATOR USING JACK-KNIFE METHOD OF ESTIMATION

Jharna Banerjie [1] and Neeraj Tiwari [2]
[1]Department of Statistics, D.A.V. (P.G.) College, Dehradun, India
E-mail: jharna_banerjie@yahoo.co.in
[2]Department of Statistics, Kumaon University, S.S.J. Campus, Almora, India
E-mail: kumarn_amo@yahoo.com

Abstract

In this paper, we propose to use an improved sampling strategy based on the modified ratio estimator using the population coefficient of variation and the coefficient of kurtosis of the auxiliary variable by Upadhyay and Singh (1999) for estimating the population mean (total) of the study variable in a finite population. Also the proposed sampling strategy is shown to be better in the sense of unbiased and smaller mean square error. A generalized Jack-Knife estimator is proposed and it is shown that the proposed Jack-Knife estimator is unbiased to the first order of approximation. A comparative study is made with usual sampling strategies utilizing the optimizing value of the characterizing scalar.

Keywords: Ratio estimator, coefficient of variation, unbiasedness, mean square error, Jack-Knife technique.

1. Introduction

The ratio estimator for estimating the population mean of the study variable *y* is given by $\bar{y}_R = \hat{R}\bar{X}$ where $\hat{R} = \bar{y}/\bar{x}$, $\bar{y}$ is the sample mean of the study variable *y*, $\bar{x}$ is the sample mean of some auxiliary variable *x* and $\bar{X}$ is the population mean of the *x* which is assumed to be known. If the population coefficient of variation of *x* denoted by C_x and population coefficient of kurtosis denoted by β_{2x} is known, then Upadhyay and Singh (1999) proposed a modified ratio estimator for estimating the population mean $\bar{Y}$ of the study variable given by

$$\bar{y}_{US} = \bar{y}\frac{\bar{X}C_x + \beta_{2x}}{\bar{x}C_x + \beta_{2x}} \tag{1.1}$$

We propose a modified ratio estimator for estimating the population mean $\bar{Y}$ of the study variable given by

$$\bar{y}_S = \bar{y}\frac{\bar{X}C_x + 1}{\bar{x}C_x + 1}$$

Our motivation of this paper is to propose an improved sampling strategy which is better in the sense of unbiasedness and smaller mean square error.

The use of population coefficient of variation has been discussed by Searls (1964), Murthy (1967), Sen (1978), Pandey & Dubey (1988) among others.

The bias and mean square error of $\bar{y}_S$ under simple random sampling are given by

$$Bias(\bar{y}_S) = \frac{1-f_n}{n}\bar{Y}(\alpha^2 C_x^2 - \rho\ \alpha\ C_x C_y)$$
$$= \gamma_n \bar{Y}(\alpha^2 C_x^2 - \rho\ \alpha\ C_x C_y) \tag{1.2}$$

and

$$MSE(\bar{y}_S) = \frac{1-f_n}{n}\bar{Y}^2\{(C_y^2 + \alpha C_x^2(\alpha - 2K)\}$$
$$= \gamma_n \bar{Y}^2\{(C_y^2 + \alpha C_x^2(\alpha - 2K)\} \tag{1.3}$$

where $f_n = \dfrac{n}{N}$ is the sampling fraction

$\gamma_n = \dfrac{1-f_n}{n}$, $\quad \alpha = \dfrac{\bar{X}C_x}{\bar{X}C_x + 1}$, $\quad K = \rho\dfrac{C_y}{C_x}$, where C_y is population coefficient of variation of y and ρ is a population correlation coefficient between *x* and *y*.

Now we propose some improved sampling strategies such that the estimator of the population mean $\bar{Y}$ is

$$\bar{y}_S = \bar{y}_s \frac{\bar{X}C_x + 1}{A(\bar{x}_s C_x + 1) + (1-A)(\bar{X}C_x + 1)} \tag{1.4}$$

We now consider this estimator under simple random sampling without replacement alongwith Jack-Knife technique and denote the resulting estimator as $\hat{\bar{Y}}_{SS}^{*}$.

2. Bias and MSE of $\bar{y}_S$ under SRSWOR alongwith Jack-Knife Technique

Consider estimator $\bar{y}_S$ under simple random sampling without replacement and denote it by $\bar{y}_{SS}$

Let $\bar{y}_s = \bar{Y} + e_0$ and $\bar{x}_s = \bar{X} + e_1$

such that

$$E(e_0) = E(e_1) = 0 \tag{2.1}$$

putting these values in the estimator

$$\bar{y}_{SS} = \bar{y}_s \frac{\bar{X}C_x + 1}{A(\bar{x}_s C_x + 1) + (1-A)(\bar{X}C_x + 1)}$$
$$= (\bar{Y} + e_0)\frac{\bar{X}C_x + 1}{A(\bar{X}C_z + e_1 C_x + 1) + (1-A)(\bar{X}C_x + 1)}$$
$$= (\bar{Y} + e_0)\left\{1 + \frac{AC_x e_1}{\bar{X}C_x + 1}\right\}^{-1}$$

$$= (\bar{Y} + e_0)\left\{1 - \frac{AC_x e_1}{\bar{X}C_x + 1} + \frac{A^2 C_x^2 e^2{}_1}{(\bar{X}C_x + 1)^2} - ...\right\}$$

$$= (\bar{Y} + e_0) - \frac{AC_x \bar{Y} e_1}{\bar{X}C_x + 1} + \frac{A^2 C_x^2 \bar{Y} e_1^2}{(\bar{X}C_x + 1)^2} - \frac{AC_x e_0 e_1}{(\bar{X}C_x + 1)} + ...$$

$$\bar{y}_{SS} - \bar{Y} = e_0 - \frac{AC_x \bar{Y} e_1}{(\bar{X}C_x + 1)} + \frac{A^2 C_x^2 \bar{Y} e_1^2}{(\bar{X}C_x + 1)^2} - \frac{AC_x e_0 e_1}{(\bar{X}C_x + 1)} \tag{2.2}$$

Taking expectation on both sides and using (2.1), we have

$$Bias\ \ \bar{y}_{SS} \ = E(\bar{y}_{SS}) - \bar{Y}$$

$$= \bar{Y}\left\{\frac{A^2 C_x^2}{(\bar{X}C_x + 1)^2} E(e_1^2) - \frac{A}{(\bar{X}C_x + 1)} E(e_0 e_1)\right\}$$

First two terms are zero as $E(e_0) = E(e_1) = 0$

Since

$$E(e_0^2) = \left(\frac{1}{n} - \frac{1}{N}\right)\bar{Y}^2 C_y^2$$

$$E(e_1^2) = \left(\frac{1}{n} - \frac{1}{N}\right)\bar{X}^2 C_x^2$$

$$E(e_0 e_1) = \left(\frac{1}{n} - \frac{1}{N}\right)\bar{X}\bar{Y}\rho C_x C_y \tag{2.3}$$

Therefore

$$Bias\ \ \bar{y}_{SS} = \bar{Y}\left(\frac{1}{n} - \frac{1}{N}\right)\left\{A^2 \alpha^2 C_x^2 - A\alpha\rho C_x C_y\right\}$$

$$= \gamma_n \bar{Y}\left\{A^2 \alpha^2 C_x^2 - A\alpha\rho C_x C_y\right\} \tag{2.4}$$

Now for mean square error, consider (2.2), upto the first order of approximation

$$MSE\ \ (\bar{y}_{SS}) = E[(\bar{y}_{SS} - \bar{Y}]^2$$

$$= E\left[e_0 - \frac{AC_x \bar{Y} e_1}{\bar{X}C_x + 1}\right]^2$$

$$= E(e_0^2) + \frac{A^2 C_x^2 \bar{Y}^2}{(\bar{X}C_x + 1)^2} E(e_1^2) - \frac{2AC_x \bar{Y}}{(\bar{X}C_x + 1)} E(e_0 e_1)$$

$$= \bar{Y}^2 \frac{N-n}{Nn}\left\{C_y^2 + A^2 \alpha^2 C_x^2 - 2A\alpha\rho C_x C_y\right\}$$

$$= \bar{Y}^2 \gamma_n \left\{C_y^2 + A^2 \alpha^2 C_x^2 - 2A\alpha\rho C_x C_y\right\}$$

$$= \bar{Y}^2 \gamma_n \left\{C_y^2 + \alpha C_x^2 (A^2 \alpha - 2AK)\right\} \tag{2.5}$$

The optimizing value of the characterizing scalar A is given by

$$A = K/\alpha = A_{opt} \text{ (say)} \tag{2.6}$$

and the minimum mean square error under optimizing value of $A = A_{opt}$ is

$$MSE(\bar{y}_{SS}) = \gamma_n \bar{Y}^2 (1-\rho^2) C_y^2 \tag{2.7}$$

which is same as the mean square error of the linear regression estimator. Also $Bias(\bar{y}_{SS}) = 0$ under the optimizing value of A.

3. Bias and MSE of Jack-Knife estimator $\hat{\bar{Y}}_{SS}^*$ under SRSWOR.

Let us now apply Quenouille's (1956) method of Jack-Knife such that the sample of size *n=2m* from a population *N* is split up at random into two sub samples of size *m* each. For further details one may refer to Gray and Schucancy (1972).

Let us define

$$\bar{y}_{ss}^{(i)} = \bar{y}_i \frac{\bar{X}C_x + 1}{A(\bar{x}_i C_x + 1) + (1-A)(\bar{X}C_x + 1)}; \qquad i=1, 2$$

$$\bar{y}_{ss}^{(3)} = \bar{y}_s \frac{\bar{X}C_x + 1}{A(\bar{x}_s C_x + 1) + (1-A)(\bar{X}C_x + 1)} \tag{3.1}$$

Where A is the characterizing scalar to be chosen suitably such that $s=s_1+s_2$, s_1 and s_2 be the two sub samples of size m each and + denotes the disjoint union. $\bar{y}_1, \bar{y}_2$ and $\bar{y}_s$ denote the sample means based on two sub samples of size m and the entire sample of size *n=2m* for characteristic y. $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_s$ denote the sample means based on two sub samples of size *m* and the entire sample of size *n=2m* for characteristic *x*.

Also, denoting $f_m = \dfrac{m}{N}$ and $\gamma_m = \dfrac{1-fm}{m}$,

it can be easily seen that

$$\text{Bias } (\bar{y}_{ss}^{(i)}) = \gamma_m \bar{Y}\left\{A^2\alpha^2 C_x^2 - A\alpha\rho C_x C_y\right\} \qquad i=1, 2$$

$$\text{Bias } (\bar{y}_{ss}^{(3)}) = \gamma_n \bar{Y}\left\{A^2\alpha^2 C_x^2 - A\alpha\rho C_x C_y\right\} \qquad = B_1 \text{ (Say)} \tag{3.2}$$

Let us define

$$\hat{\bar{Y}}'_{SS} = \frac{\bar{y}_{SS}^{(1)} + \bar{y}_{SS}^{(2)}}{2}$$

as an alternative estimator of the population mean $\bar{Y}$.

The bias of $\hat{\bar{Y}}'_{SS}$ is

$$\text{Bias } (\hat{\bar{Y}}'_{SS}) = \gamma_m \bar{Y}\left\{A^2\alpha^2 C_x^2 - A\alpha\rho C_x C_y\right\} = B_2 \text{ (Say)} \tag{3.3}$$

We propose the jack the Jack-Knife estimator $\hat{\bar{Y}}_{SS}^*$ for estimating population mean $\bar{Y}$ given by

$$\hat{\bar{Y}}_{SS}^{*} = \frac{\bar{y}_{SS}^{(3)} - R\hat{\bar{Y}}_{SS}^{'}}{1-R} \text{ where } R = \frac{B_1}{B_2}$$

$$= \frac{\bar{y}_{SS}^{(3)} - \left\{\frac{N-2n}{2(N-n)}\right\}\hat{\bar{Y}}_{SS}^{'}}{1-\left\{\frac{N-2n}{2(N-n)}\right\}} \tag{3.4}$$

Taking expectation of (3.4) and using (3.2) and (3.3) we get.

$E(\hat{\bar{Y}}_{SS}^{*}) = \bar{Y}$ showing that $\hat{\bar{Y}}_{SS}^{*}$ is an unbiased estimator of population mean $\bar{Y}$ to the first order of approximation.

consider

$$MSE(\hat{\bar{Y}}_{SS}^{*}) = E\left\{\hat{\bar{Y}}_{SS}^{*} - \bar{Y}\right\}^2$$

$$= E\left\{\frac{\bar{y}_{SS}^{(3)} - R\hat{\bar{Y}}_{SS}^{'}}{1-R} - \bar{Y}\right\}^2$$

$$= \frac{1}{(1-R)^2} E\left\{\bar{y}_{SS}^{(3)} - R\hat{\bar{Y}}_{SS}^{1} - \bar{Y} + R\bar{Y}\right\}^2$$

$$= \frac{1}{(1-R)^2} E\left\{(\bar{y}_{SS}^{(3)} - \bar{Y}) - R(\hat{\bar{Y}}_{SS}^{'} - \bar{Y})\right\}^2$$

$$= \frac{1}{(1-R)^2}\left\{E(\bar{y}_{SS}^{(3)} - \bar{Y})^2 + R^2 E(\hat{\bar{Y}}_{SS}^{'} - \bar{Y}) - 2RE(y_{SS}^{(3)} - \bar{Y})(\hat{\bar{Y}}_{SS}^{'} - \bar{Y})\right\} \tag{3.5}$$

Also

$$E(\bar{y}_{SS}^{(3)} - \bar{Y})^2 = MSE(\bar{y}_{SS}^{(3)})$$

$$= \gamma_n \bar{Y}^2\left\{C_y^2 + \alpha^2 C_x^2 (A^2\alpha - 2AK)\right\} \tag{3.6}$$

Further

$$E(\hat{\bar{Y}}_{SS}^{'} - \bar{Y})^2 = E\left[\frac{\bar{y}_{SS}^{(1)} + \bar{y}_{SS}^{(2)}}{2} - \bar{Y}\right]^2$$

$$= \frac{1}{4}E\left\{(\bar{y}_{SS}^{(1)} - \bar{Y}) + (\bar{Y}_{SS}^{(2)} - \bar{Y})\right\}^2$$

$$= \frac{1}{4}\left\{E(\bar{y}_{SS}^{(1)} - \bar{Y})^2 + E(\bar{y}_{SS}^{(2)} - \bar{Y})^2 + 2E(\bar{y}_{SS}^{(1)} - \bar{Y})(\bar{y}_{SS}^{(2)} - \bar{Y})\right\} \tag{3.7}$$

since

$$E(\bar{y}_{SS}^{(1)} - \bar{Y})^2 = MSE(\bar{y}_{SS}^{(1)})$$

$$= \gamma_m \bar{Y}^2 \left\{ C_y^2 + \alpha C_x^2 (A^2\alpha - 2AK) \right\}$$

$$E(\bar{y}_{SS}^{(2)} - \bar{Y})^2 = MSE(\bar{y}_{SS}^{(2)})$$

$$= \gamma_m \bar{Y}^2 \left\{ C_y^2 + \alpha C_x^2 (A^2\alpha - 2AK) \right\} \qquad (3.8)$$

Let

$\bar{y}_i = \bar{Y} + e_0^{(i)}$ and $\bar{x}_i = \bar{X} + e_1^{(i)}$ such that

$E(e_0^{(i)} = E(e_1^{(i)}) = 0$; i=1, 2

Consider,

$$E(\bar{y}_{SS}^{(1)} - \bar{Y})(\bar{y}_{SS}^{(2)} - \bar{Y}) = E\left[\left\{ e_0^{(1)} - \frac{A\bar{Y}e_1^{(1)}}{\bar{X}C_x + 1} \right\} \left\{ e_0^{(2)} - \frac{A\bar{Y}e_1^{(2)}}{\bar{X}C_x + 1} \right\} \right]$$

$$= E\left\{ e_0^{(1)}.e_0^{(2)} \right\} - \frac{AC_x\bar{Y}}{(\bar{X}C_x + 1)} \left\{ E(e_0^{(2)}e_1^{(1)} + E(e_0^{(1)}.e_1^{(2)} \right\}$$

$$+ \left\{ \frac{AC_x\bar{Y}}{(\bar{X}C_x + 1)} \right\}^2 E\left(e_1^{(1)}e_1^{(2)} \right)$$

Substituting the results in Sukhatme and Sukhatme (1997)

$$E(e_0^{(1)}e_0^{(2)}) = -\frac{1}{N}\bar{Y}^2 C_y^2$$

$$E(e_1^{(1)}e_1^{(2)}) = -\frac{1}{N}\bar{X}^2 C_x^2$$

$$E(e_0^{(1)}e_1^{(2)}) = E(e_0^{(2)}e_1^{(1)}) = -\frac{1}{N}\overline{XY}\rho C_x C_y$$

We have

$$E(\bar{y}_{SS}^{(1)} - \bar{Y})(\bar{y}_{SS}^{(2)} - \bar{Y}) = -\frac{1}{N}\bar{Y}^2 \left\{ C_y^2 + A^2\alpha^2 C_x^2 - 2A\alpha\rho C_x C_y \right\}$$

$$= -\frac{1}{N}\bar{Y}^2 \left\{ C_y^2 + \alpha C_x^2 (A^2\alpha - 2AK) \right\} \qquad (3.9)$$

Putting values from (3.8) and (3.9) in (3.7) we have

$$E(\hat{\bar{Y}}_{SS}^{'} - \bar{Y})^2 = \bar{Y}^2.\frac{1}{4}(2\gamma_m - \frac{2}{N})\left\{ C_y^2 + \alpha C_x^2 \left(A^2\alpha - 2AK \right) \right\}$$

$$= \gamma_n \bar{Y}^2 \left\{ C_y^2 + \alpha C_x^2 (A^2\alpha - 2AK) \right\} \qquad (3.10)$$

Now

$$E\left(\bar{y}_{SS}^{(3)}-\bar{Y}\right)\left(\hat{\bar{Y}}_{SS}^{'}-\bar{Y}\right)=E\left\{(\bar{y}_{SS}^{(3)}-\bar{Y})\left(\frac{\bar{y}_{SS}^{(1)}+\bar{y}_{SS}^{(2)}}{2}-\bar{Y}\right)\right\}$$

$$=\frac{1}{2}\left\{E(\bar{y}_{SS}^{(3)}-\bar{Y})(\bar{y}_{SS}^{(1)}-\bar{Y})+E(\bar{y}_{SS}^{(3)}-\bar{Y})(\bar{y}_{SS}^{(2)}-\bar{Y})\right\}$$

Since

$$E(\bar{y}_{SS}^{(3)}-\bar{Y})(\bar{y}_{SS}^{(i)}-\bar{Y})=E\left[\left\{e_0-\frac{AC_x\bar{Y}e_1}{(\bar{X}C_x+1)}\right\}\left\{e_0^{(i)}-\frac{AC_x\bar{Y}e_1^{(i)}}{(\bar{X}C_x+1)}\right\}\right];\ i=1,2$$

$$=E(e_0e_0^{(i)})-\frac{AC_x\bar{Y}}{(\bar{X}C_x+1)}\left\{E(e_0^{(i)}e_1)+E(e_0e_1^{(i)}\right\}+\left(\frac{AC_x\bar{Y}}{\bar{X}C_x+1}\right)^2E(e_1e_1^{i})$$

Using the following results given in Sukhatme and Sukhatme(1997)

$$E(e_0e_0^{(i)})=\gamma_n\bar{Y}^2C_y^2$$

$$E(e_1e_1^{(i)})=\gamma_n\bar{X}^2C_x^2$$

$$E(e_0e_1^{(i)})=E(e_0^{(i)}e_1)=\gamma_n\bar{X}\bar{Y}\rho C_xC_y\quad;\qquad i=1,2$$

We have

$$E(\bar{y}_{SS}^{(3)}-\bar{Y})(\bar{y}_{SS}^{(i)}-\bar{Y})=\gamma_n\bar{Y}^2\left\{C_y^2+\alpha C_x^2(A^2\alpha-2AK)\right\}\tag{3.11}$$

Putting these values from (3.6), (3.10) and (3.11) in (3.5) we have

$$MSE(\hat{\bar{Y}}_{SS}^{*})=\gamma_n\bar{Y}^2\frac{1}{(1-R)^2}(1+R^2-2R)\left\{C_y^2+\alpha C_x^2(A^2\alpha-2AK)\right\}$$

$$=\gamma_n\bar{Y}^2\left\{C_y^2+\alpha C_x^2(A^2\alpha-2AK)\right\}\tag{3.12}$$

Which is equal to $MSE(\bar{y}_{SS})$

The optimizing value of the characterising scalar A is given by

$$A=\frac{K}{\alpha}=A_{opt}$$

The minimum *MSE* under optimizing value of a = A_{opt} is

$$MSE(\hat{\bar{Y}}_{SS}^{*})=\gamma_n\bar{Y}^2(1-\rho^2)C_y^2\tag{3.13}$$

which is same as the mean square error of the linear regression estimator.

4. Estimation under the Estimated Value of the Characterizing Scalar

The mean square error of the estimator $\bar{y}_{SS}$ is minimized for the choice of $A=\dfrac{K}{\alpha}=A_{opt}$ and the minimum mean square error of $\bar{y}_{SS}$ is given by $MSE(\bar{y}_{SS})=\gamma_n\bar{Y}^2(1-\rho^2)C_y^2$.

Since the estimated value of the characterizing scalar *A* may not be known in practice as it may involve unknown parameters; then we may estimate it by replacing the unknown parameters by their unbiased estimators. The optimizing value of the unknown parameter *A* can be written as

$$A_{opt} = \frac{K}{\alpha} = \frac{1}{\alpha}\frac{\bar{X}}{\bar{Y}}\frac{S_{xy}}{S_x^2}$$

Let $\hat{A}$ denote the estimator of the optimizing value of the characterizing scalar A_{opt} given by

$$\hat{A} = \frac{\hat{K}}{\alpha} = \frac{1}{\alpha}\frac{\bar{X}}{\bar{y}}\frac{s_{xy}}{S_x^2} \tag{4.1}$$

Now the estimator under the estimated value of the optimizing scalar $\hat{A}$ takes the following form

$$\bar{y}_{\hat{A}} = \bar{y}\frac{\bar{X}C_x + 1}{\hat{A}(\bar{x}C_x + 1) + (1-\hat{A})(\bar{X}C_x + 1)} \tag{4.2}$$

In order to obtain the mean square error of $\hat{\bar{Y}}_\theta$, let us denote by

$$\bar{y} = \bar{Y} + e_0$$
$$\bar{x} = \bar{X} + e_1$$
$$s_{xy} = S_{xy} + e_2$$

with $\quad E(e_0) = E(e_1) = E(e_2) = 0 \qquad (4.3)$

Putting these values in (4.2) we have

$$\bar{y}_{\hat{A}} = \bar{y}\frac{\bar{X}C_x + 1}{\hat{A}(\bar{x}C_x + 1) + (1-\hat{A})(\bar{X}C_x + 1)}$$
$$= \bar{y}\frac{\bar{X}C_x + 1}{\bar{X}C_x + 1 + \hat{A}(\bar{x} - \bar{X})C_x}$$
$$= (\bar{Y} + e_0)\frac{\bar{X}C_x + 1}{\bar{X}C_x + 1 + \frac{1}{\alpha}\frac{\bar{X}}{\bar{Y} + e_0}\frac{S_{xt} + e_2}{S_x^2}(\bar{x} - \bar{X})C_x}$$
$$= (\bar{Y} + e_0)\frac{\bar{X}C_x + 1}{\bar{X}C_x + 1 + A_{opt}e_1\left(1 + \frac{e_0}{\bar{Y}}\right)^{-1}\left(1 + \frac{e_2}{S_{xy}}\right)C_x}$$

$$= \left(\bar{Y} + e_0\right)\left\{1 + \frac{A_{opt}e_1\left(1+\frac{e_0}{\bar{Y}}\right)^{-1}\left(1+\frac{e_2}{S_{xy}}\right)C_x}{\bar{X}C_x + 1}\right\}^{-1}$$

$$= \left(\bar{Y} + e_0\right)\left\{1 + \frac{A_{opt}e_1\left(1-\frac{e_0}{\bar{Y}}+\ldots\right)^{-1}\left(1+\frac{e_2}{S_{xy}}\right)C_x}{\bar{X}C_x + 1} + \ldots\right\}$$

$$\bar{y}_{\hat{A}} - \bar{Y} = e_0 - \frac{\bar{Y}A_{opt}C_x}{\left(\bar{X}C_x + 1\right)}e_1 + O(e^2)$$

Taking expectation on both sides and using (4.3) we have

$$MSE(\bar{y}_{\hat{A}}) = E(\bar{y}_{\hat{A}} - \bar{Y})^2$$

$$= E\left\{e_0 - \frac{\bar{Y}A_{opt}C_x}{\left(\bar{X}C_x + 1\right)}e_1\right\}^2$$

$$= E(e_0^2) + \frac{\bar{Y}^2 A_{opt}^2 C_x^2}{(\bar{X}C_x + 1)^2}E(e_1^2) - 2\frac{\bar{Y}A_{opt}C_x}{(\bar{X}C_x + 1)}e_1 E(e_0 e_1)$$

$$= \bar{Y}^2\gamma_n\left(\frac{N-n}{N.n}\right)\left\{C_y^2 + A_{opt}^2\alpha^2 C_x^2 - 2A_{opt}\alpha\rho C_x C_y\right\}$$

$$= \bar{Y}^2\gamma_n\left\{C_y^2 + \alpha C_x^2(A_{opt}^2\alpha - 2A_{opt}K)\right\}$$

$$= \bar{Y}^2\gamma_n(1-\rho^2)C_y^2$$

$$= MSE(\bar{y}_{SS})_{\min} \tag{4.4}$$

which shows that the estimator based on estimated value of the characterizing scalar attains the minimum mean square error in the class of all estimators.

5. Comparison with Some Commonly Used Estimators

The proposed estimator is almost unbiased and under the optimizing value of the characterizing scalar attains its minimizing value given by

$$MSE(\bar{y}_{SS})_{\min} = \gamma_n\bar{Y}^2(1-\rho^2)C_y^2$$

$$= MSE(\bar{y}_{\hat{A}}) = P(say) \tag{5.1}$$

The ratio estimator $\bar{y}_R$ of the population mean $\bar{Y}$ is a biased estimator whose bias and mean square error are given by

$$Bias(\bar{y}_R) = \gamma_n \bar{Y}(C_x^2 - \rho\ C_x C_y) \tag{5.2}$$

$$MSE(\bar{y}_R) = \gamma_n \bar{Y}^2 \left\{C_y^2 + C_x^2 - 2\rho\ C_x C_y\right\} \tag{5.3}$$

Similarly product estimator $\bar{y}_P$ of the population mean $\bar{Y}$ is also a biased estimator and its bias & mean square error is given by

$$Bias(\bar{y}_P) = \gamma_n \bar{Y}(C_x^2 + \rho\ C_x C_y) \tag{5.4}$$

$$MSE(\bar{y}_P) = \gamma_n \bar{Y}^2 \left\{C_y^2 + C_x^2 + 2\rho\ C_x C_y\right\} \tag{5.5}$$

Also linear regression estimator of the population mean $\bar{Y}$ is a biased estimator and its mean square error, upto the first order of approximation, is given by

$$MSE(\bar{y}_{lr}) = \gamma_n \bar{Y}^2 (1-\rho^2) C_y^2 \tag{5.6}$$

Now comparing (5.1) with (5.3), (5.5) and (5.6), we have

$$MSE(\bar{y}_R) - P = \gamma_n \bar{Y}^2 (\rho\ C_y - C_x)^2 \geq 0 \tag{5.7}$$

$$MSE(\bar{y}_P) - P = \gamma_n \bar{Y}^2 (\rho\ C_y + C_x)^2 \geq 0 \tag{5.8}$$

$$MSE(\bar{y}_{lr}) - P = 0 \tag{5.9}$$

6. Empirical Study

Let us consider the following example considered by Singh & Chaudhary (1986) wherein the following values were obtained:

$\bar{Y}$ =22.62, $\bar{X}$ =1467.55, C_x =1042.46, C_y =1.7459,

β_{2x} =5.5788 and ρ =0.9022

The bias and M.S.E. of sample mean $\bar{y}$, ratio estimator $\bar{y}_R$, product estimator $\bar{y}_P$, Upadhyay & Singh $\bar{y}_{US}$, and linear regression $\bar{y}_{lr}$ are given in the following table.

Estimators	**Bias**	**M.S.E.**
$\bar{y}$	0	γ_n (6564590)
$\bar{y}_R$	γ_n (-0.1667)	γ_n (0.5804)
$\bar{y}_P$	γ_n (4.4352)	γ_n (9.7843)
$\bar{y}_{lr}$	γ_n (3867.58)	γ_n (0.5673)
$\bar{y}_{US}$	γ_n (-0.2782)	γ_n (0.6085)
$\hat{\bar{Y}}_{SS}^*$	0	γ_n (0.5673)

Table: Comparison of Bias and M.S.E. for various estimators

This study shows the proposed sampling strategy is better than other sampling strategies in terms of unbiasedness and lesser M.S.E.

7. Concluding Remarks

From the above expressions, we conclude that the proposed sampling strategy under the optimizing value of the characterizing scalar is always more efficient than that of ratio estimator, product estimator whereas it is equally efficient as linear regression estimator. Thus the proposed estimator under Jack-Knife technique is unbiased upto first order of approximation.

References

1. Gray H. L. and Schucany W.R. (1972). The Generalized Jack-knife Statistic, Marcel Dekker, New York.
2. Murthy M.N. (1967). Sampling Theory and Methods. Statistical Publishing Society, Calcutta, India.
3. Pandey B. N. and Dubey V. (1988). Modified product estimator using coefficient of variation of auxiliary variate, Assam Stat. Rev. 2(2) (1988), p. 64-66.
4. Quenouille M.H. (1956). Notes on bias in estimation, Biometrika, 43, p. 353-360.
5. Searls D.T. (1964). The utilization of known coefficient of variation in the estimation procedure, Jour. Amer. Stat. Asso. 59 , p. 1125-26.
6. Sen A.R. (1978). Present status of probability sampling and its use in the estimation, Stat. Theory Method, A7, p. 657-672.
7. Singh D. and Chaudhary F.S. (1986). Theory and Analysis of Sampling Survey Design, Wiley Eastern Limited, New Delhi.
8. Sukhatme P.V. and Sukhatme B.V. (1977). Sampling Theory of Surveys with Applications, Piyush Publications, Delhi.
9. Upadhyay, L.N. and Singh, H.P. (1999). Use of transformed auxiliary variable in estimating the finite population mean, Biometrical Journal, 41(5), p. 627-636.