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STOCHASTIC MODELING AND PERFORMANCE ANALYSIS OF A 2(K)-OUT-OF-3(N) SYSTEM WITH INSPECTION SUBJECT TO OPERATIONAL RESTRICTION

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Abstract

In repairable redundant systems the failed units can either be repaired or replaced by identical standby to reduce the system down time. The failed units are inspected for repair/replacement. In this paper, two stochastic models for 2(k)-out-of-3(n) redundant system of identical units with repair and inspection are examined stochastically. The system is considered in up-state only if 2(k)-out-of-3(n) units are operative in both the models. Normally, the server either attends the system promptly or may take some time, after failure. The system is studied under an operational restriction on the inspection i.e. in case when system has only one unit in operational mode the server has to attend the system for inspection. Semi-Markov processes and Regenerative point technique is adopted to obtain the expressions for measures of system effectiveness such as transition probabilities, mean sojourn times, mean time to system failure, steady state availabilities, busy periods, expected number of visits etc. Cost-analysis is also carried out for the system models.

Key words: Stochastic modeling, performance analysis, operational restriction, arbitrary distribution, re-generative point, semi-Markov process.

1. Introduction

Redundancy techniques are widely used to improve system performance in terms of reliability and availability. Among various redundancy techniques standby is the simplest and commonly accepted one. In general there are three types of standby; cold, warm and hot standby. Hot standby implies that the redundant (spare) unit or component has same failure rate as when it is in operation mode where as in case of cold standby the failure rate of the redundant unit or component is zero and it can't fail in standby mode. Between hot and cold there is an intermediate case known as warm standby. In this case the failure rate of redundant unit lies in between that of hot and cold standby.

In order to reduce the down time redundancy is necessary. In literature, many researchers have been discussed the reliability and availability of standby systems in detail by considering different cases and strategies such as by considering weather conditions [2], replacement policy with spares [3], random appearance-disappearance time of service facility [4], preventively maintained identical units [5], general distributions [6], correlated failures and repair [7], multiple critical errors [8], dissimilar unit system with perfect or imperfect switch [9], complex standbys (cold→warm, warm→cold etc) [10]. Further, Yadvalli et al. [11] dealt with asymptotic confidence limits for the steady state availability of a two-unit parallel system with preparation time for the repair facility. Edmond et al. [12] carried out reliability analysis of a renewable multiple cold standby system. Some recent related and extended text is reported in [13-15].

In this paper two probabilistic models of a 2(k)-out-of-3(n) cold standby system are examined stochastically. Such systems found applications in various fields including the process industry, power plants, airline companies, medical diagnosis, network design and many more. For such a system when an operating unit fails the standby unit becomes operative and the system works if at least 2(k)-out-of-3(n) units are in operative mode. In model I, server attends the system promptly whenever needed and first inspects the failed unit to see the practicability of its repair. If repair of the unit is not practicable, it is replaced by new one so that unnecessary expanses on repair can be avoided. In real life, it is not always possible for the server to attend the system swiftly when required may because of his pre-occupation. In such a situation server may be allowed to take some time to reach the system. But it is urgently required that the server must arrive at the system promptly in case of urgent situation. In model II, the server takes some time to arrive at the system when 2(k)-out-of-3(n) units are operative. While in case when the system has only one unit in operational mode the server has to attend the system swiftly for inspection due to operational restriction imposed on it, so that the down time of the system may be reduced. The switches are perfect and instantaneous. Failure time follows negative exponential distribution while repair and inspection times follow arbitrary distributions. All the random variables are mutually independent and un-correlated. The expressions for various measures of system performances such as transition probabilities, sojourn times, MTSF, availability, busy period of server, expected number of visits and profit function are drawn for steady state.

2. Notations

No	Units in normal mode and operative
$\overline{N}_0$	Units in normal mode but not working
S_i	ith transition state
Cs	Unit in normal mode and cold standby
a/b	Probability that repair is useful / not useful
λ	Constant failure rate of an operative unit.
$q_{ij}(t)$ / $Q_{ij}(t)$	pdf / cdf of first passage time from a regenerative state i to a regenerative state j or to a failed state without visiting any other regenerative state in (0,t].
$q_{ij.kr}(t)/Q_{ij.kr}(t)$	pdf / cdf of first passage time from a regenerative state i to regenerative state j or to a failed state j visiting states k, r once in (0,t].
h(t)/H(t)	pdf / cdf of inspection time

w(t)/W(t)	pdf /cdf of waiting time of the server to arrive at the system.
g(t)/G(t)	pdf / cdf of repair time of the server.
$F_{wi}/F_{wI}/F_{ui}/F_{uI}$	Unit is completely failed and waiting for inspection / waiting for inspection continuously from previous state/ under inspection / under continuous inspection from previous state.
Fuii	Failed unit under immediate/ urgent inspection.
F_{ur}/ F_{UR}	Unit is completely failed and under repair / under repair continuously from previous state.
$p_{ij}/ p_{ij.kr}$	Probability of transition from regenerative state i to a regenerative state j without visiting any other state in (0,t] / visiting state k,r once in (0,t] i.e. $p_{ij} = \lim_{s\to 0} q_{ij}^{*}(s)$ and $p_{ij.kr} = \lim_{s\to 0} q_{ij.kr}^{*}(s)$
* / $\boxed{s}$	Laplace / Laplace-Stiltje's transform.

3. Transition states

The following are the possible transition states of the system.

For model-I

$S_0 = \{N_0, N_0, C_s\}, S_1 = \{N_0, N_0, F_{ui}\}, S_2 = \{\overline{N}_0, F_{wi}, F_{UI}\}, S_3 = \{N_0, N_0, F_{ur}\}, S_4 = \{\overline{N}_0, F_{wi}, F_{UR}\}, S_5 = \{\overline{N}_0, F_{WI}, F_{ur}\}$

The states S_0, S_1, S_3 are regenerative states while states S_2, S_4, S_5 are failed and non-regenerative states.

For model-II

$S_0 = \{N_0, N_0, C_s\}, S_1 = \{N_0, N_0, F_{wi}\}, S_2 = \{N_0, N_0, F_{ui}\}, S_3 = \{N_0, N_0, F_{ur}\}, S_4 = \{\overline{N}_0, F_{wi}, F_{uii}\},$

$S_5 = \{\overline{N}_0, F_{WI}, F_{ur}\}, S_6 = \{\overline{N}_0, F_{wi}, F_{UI}\}, S_7 = \{\overline{N}_0, F_{wi}, F_{UR}\}$

The states S_0, S_1, S_2, S_3, S_4 are regenerative states while states S_5, S_6, S_7 are non-regenerative as well as failed states. S_4 is regenerative but failed.

4. Transition probabilities

Simple probabilistic considerations yield the following expressions for the non-zero elements

$$p_{ij} = Q_{ij}(\infty) = \int q_{ij}(t)\,dt \text{ as} \tag{4.1}$$

For model-I

$p_{01} = 1, p_{10} = bh^{*}(2\lambda), p_{12} = 1 - h^{*}(2\lambda), p_{13} = ah^{*}(2\lambda), p_{30} = g^{*}(2\lambda), p_{34} = \{-g^{*}(2\lambda)\},$

$p_{11.2} = b\{1 - h^*(2\lambda)\}, p_{11.25} = a\{1 - h^*(2\lambda)\}, p_{31.4} = p_{34}$ (4.2)

It can be easily verified that

$p_{01} = 1 = p_{10} + p_{12} = p_{30} + p_{34} = p_{30} + p_{31.4} = p_{10} + p_{13} + p_{11.2} + p_{11.25}$

For model-II

$p_{12} = 1, p_{12} = w^*(2\lambda), p_{14} = 1 - w^*(2\lambda), p_{20} = bh^*(2\lambda), p_{23} = ah^*(2\lambda), p_{26} = 1 - h^*(2\lambda),$

$p_{22.65} = a\{1 - h^*(2\lambda)\}, p_{30} = g^*(2\lambda), p_{37} = 1 - g^*(2\lambda), p_{32.7} = 1 - g^*(2\lambda), p_{42} = b = p_{62},$

$p_{45} = a = p_{65}, p_{52} = p_{72} = 1, p_{42.5} = a$ (4.3)

It can be easily verified that

$p_{01} = 1 = p_{12} + p_{14} = p_{20} + p_{23} + p_{26} = p_{30} + p_{37} = p_{33} + p_{32.7}$
$= p_{42} + p_{45} = p_{52} = p_{62} + p_{65} = p_{72}$

5. Sojourn times

The unconditional mean time taken by the system to transit to any regenerative state S_i when it (time) is counted from epoch of entrance into that state, is given by

$m_{ij} = \int_0^\infty td\{Q_{ij}(t)\} = -q^{*'}_{ij}(0)$ (5.1)

The mean Sojourn time in the state S_i is given by

$\mu_i = E(t) = \int_0^\infty p(T > t)\, dt,$ T denotes the time to system failure. (5.2)

Using these, we have following expressions for mean sojourn times

For model-I

$\mu_0 = \frac{1}{2\lambda}, \mu_1 = \frac{1}{2\lambda}\{1 - h^*(2\lambda)\}, \mu_3 = \frac{1}{2\lambda}\{1 - g^*(2\lambda)\}$ (5.3)

For model-II

$\mu_0 = \frac{1}{2\lambda}, \mu_1 = \frac{1}{2\lambda}\{1 - w^*(2\lambda)\}, \mu_2 = \frac{1}{2\lambda}\{1 - h^*(2\lambda)\}, \mu_3 = \frac{1}{2\lambda}\{1 - g^*(2\lambda)\}$ (5.4)

6. MTSF Analysis

On the basis of arguments used for regenerative processes, we obtain the expressions for cdf ($\emptyset_i(t)$) of first passage times from regenerative state i to a failed states

$\emptyset_i(t) = \sum_{ij} Q_{ij}(t) \boxed{s} \emptyset_j(t)$ (6.1)

Solving above recursive differential- difference equations (6.1) for $\tilde{\emptyset}_0(s)$ using Laplace and Laplace-Stiltje's transforms and letting $t\to\infty$ i.e. $s\to 0$ we get in the long run the expected time for which the system is in operation before it completely fails as

$MTSF(T_i) = \lim_{s\to 0} R^*(s) = \lim_{s\to 0} \frac{1 - \tilde{\emptyset}_0(s)}{s} = \frac{N_{i1}}{D_{i1}}, i = 1,2\ (for\ model - I\ and\ model - II\ respectively)$ (6.2)

$N_{11} = \frac{1}{2\lambda}[2 + h^*(2\lambda)]\ and\ D_{11} = [1 - h^*(2\lambda)]$ (6.3)

$N_{21} = \frac{1}{2\lambda}[2 + w^*(2\lambda)h^*(2\lambda)\{a - 1 - ag^*(2\lambda)\}]$ and $D_{21} =$
$1 - w^*(2\lambda)h^*(2\lambda)\{b + ag^*(2\lambda)\}$ (6.4)

7. Availability analysis

7.1 Steady state availability

Let $A_i(t)$ be the probability that the system is in up-state at instant t given that the system entered regenerative state i at t=0.The recursive relations giving point-wise availability $A_i(t)$ are given as

$$A_i(t) = M_i(t) + \sum_{ij}\ [q_{ij}(t) + q_{ij.kr}(t)] \copyright A_j(t) \quad (7.1.1)$$

$M_i(t)$ represents the probability that the system is up initially in regenerative state S_i, is up at time t without passing through any other regenerative state or returning to itself through one or more non-regenerative states i.e. either it continues to remain in regenerative state S_i or in a non-regenerative state including itself.

$$for\ model - I: M_0(t) = e^{-2\lambda t}, M_1(t) = e^{-2\lambda t}\ \bar{H}, M_3(t) = e^{-2\lambda t}\ \bar{G} \quad (7.1.2)$$

$$for\ model - II: M_0(t) = e^{-2\lambda t}, M_1(t) = e^{-2\lambda t}\ \bar{W}(t), M_2(t) = e^{-2\lambda t}\ \bar{H}(t), M_3(t) = e^{-2\lambda t}\ \bar{G} \quad (7.1.3)$$

Solving above recursive relations (7.1.1) for $A_0^*(s)$ by using Laplace transform and letting t→∞ i.e. s→0 we get the asymptotic availability of the system models as

$$A_{i0}(\infty) = A_{i0} = \lim_{s\to 0} sA_0^*(s) = \frac{N_{i2}}{D_{i2}},\ i = 1,2\ (for\ model - I\ and\ model - II\ respectively) \quad (7.1.4)$$

$N_{12} = 1/2\lambda$ and $D_{12} = \frac{1}{2\lambda}\left[\{b + ag^*(2\lambda)\}h^*(2\lambda) + ah^*(2\lambda)\left\{\{1 - g^{\acute{*}}(0)\} - g^*(2\lambda)\{1 - g^{\acute{*}}(0)\}\right\}\right] - h^{\acute{*}}(2\lambda)$

$N_{22} = \frac{1}{2\lambda}[\{2 - w^*(2\lambda)\}a\{1 + h^*(2\lambda)g^*(2\lambda)\} + (a-1)h^*(2\lambda) + \{1 - g^*(2\lambda)h^*(2\lambda)\}]$ and

$D_{22} = h^*(2\lambda)[(b + a)g^*(2\lambda)\left(\frac{1}{2\lambda}\right)\{1 + w^*(2\lambda)\} + 1 + a\{1 - g^*(2\lambda)\}\left\{\frac{1}{2\lambda} - g^{\acute{*}}(0)\right\} - \{1 - w^*(2\lambda)\}\{b + ag^*(2\lambda)\}\{(1 + a)h^{\acute{*}}(0)] - h^{\acute{*}}(0) + a\{1 - h^*(2\lambda)\}\{\frac{1}{2\lambda} - g^{\acute{*}}(0) - h^{\acute{*}}(0)\}$

(7.1.5)

7.2 Interval availability

The probability that the system is available for use in interval [0,t] is given by

$$IA_{i0} = \frac{1}{t}\int_0^t A_{i0}(u)du\ \ i.e.\ IA_{i0}^*(s) = \int_s^\infty \frac{A_{i0}^*(u)}{u}du \quad (7.2.1)$$

Expected up time in (0,t] is

$$\mu_{up}(t) = \int_0^t A_{i0}(u)du \quad (7.2.2)$$

Expected downtime in (0,t] is

$$\mu_{dn}(t) = t - \mu_{up}(t) \quad (7.2.3)$$

Above values can be obtained numerically by using the concept of Laplace transform.

8. Busy Period Analysis

8.1 Expected busy period of server in long run

Let $B_i(t)$ be the probability that the server is busy at an instant time t given that the system entered the regenerative-state i at t=0.Then recursive relations for $B_i(t)$ are given as

$$B_i(t) = W_i(t) + \sum_{ij} [q_{ij}(t) + q_{ij.kr}(t)] © B_j(t) \quad (8.1.1)$$

$for\ model - I: W_1(t) =$

$$[e^{-2\lambda t} + \{2\lambda e^{-2\lambda t} ©1\}]\bar{H} + \{2\lambda e^{-2\lambda t} ©ah(t)©1\}]\bar{G}\ and$$

$$W_3(t) = [e^{-2\lambda t} + \{2\lambda e^{-2\lambda t} ©1\}]\bar{G} \quad (8.1.2)$$

$for\ model - II: W_2(t) =$

$$\left[e^{-2\lambda t} + \left(2\lambda e^{-2\lambda t} ©1\right)\right]\bar{H}(t) + \left[\left(2\lambda e^{-2\lambda t} ©ah(t)©1\right)\right]\bar{G}(t)$$

$$W_3(t) = \left[e^{-2\lambda t} + \left(2\lambda e^{-2\lambda t} ©1\right)\right]\bar{G}(t), W_4(t) = \bar{H}(t) + \{ah(t)©1\}\bar{G}(t) \quad (8.1.3)$$

$W_i(t)$ represents the probability that the server is busy in state S_i up to time t without making any transition to any other regenerative state or returning to the same via one or more non-regenerative states.

Solving above recursive relations (8.1.1) for $B_0^*(s)$ by using Laplace transform and letting t→∞ i.e. s→0 we get, asymptotically, the time for which the system is under repair as

$$B_{io}(\infty) = B_{i0} = \lim_{s\to 0} sB_0^*(s) = \frac{N_{i3}}{D_{i2}}, i = 1,2\ (for\ model - I\ and\ model - II\ respectively) \quad (8.1.4)$$

$$N_{13} = -h^{*'}(0) - ag^{*'}(0)[E_1^*(0) + h^*(2\lambda)]$$

$$N_{23} = -h^{*'}(0) - aE_1^*(0)g^{*'}(0) - \{1 - w^*(2\lambda)\}\{b + ah^*(2\lambda)g^*(2\lambda)\}\{h^{*'}(0) + aE_2^*(0)g^{*'}(0)\} \quad (8.1.5)$$

$$E_1(t) = \int_0^t 1 - e^{-2\lambda(t-u)}\, h(u)\, du\ and\ E_2(t) = \int_0^t h(u)du \quad (8.1.6)$$

8.2 Expected busy period in (0,t]

The expected busy period in (0,t] is given by

$$\mu_b(t) = \int_0^t B_{i0}(u)du \quad (8.2.1)$$

Numerical values of these time periods can be obtained for particular values of t.

9. Expected Number of Visits by the Server

Let $N_i(t)$ denotes the expected number of visits by the server in (0,t], given that the system entered the regenerative state i at t=0. The recursive relations of $N_i(t)$ are given as

$N_i(t) = \sum_{ij} Q_{ij}(t) \boxed{s} [\delta_{ij} + N_j(t)],\ s.t.\ \delta_{ij} = \begin{cases} 1; if\ there\ is\ a\ visit\ in\ transition\ from\ \ i \to j \\ 0;\ otherwise \end{cases}$ (9.1)

Solving above differential- difference equations (9.1) for $\tilde{N}_0(s)$ using Laplace and Laplace- Stiltje's transforms, and letting t→∞ i.e. s→0 we get, asymptotically, the expected number of visits per unit time of the server as

$N_{i0}(\infty) = N_{i0} = \lim_{s\to 0} s\tilde{N}_0(s) = \frac{N_{i4}}{D_{i2}}$

$i = 1,2\ (for\ model - I\ and\ model - II\ respectively)$ (9.2)

$N_{14} = [b + ag^*(2\lambda)]h^*(2\lambda), N_{24} = [b + ah^*(2\lambda)g^*(2\lambda)]w^*(2\lambda)$ (9.3)

10. Expected number of restrictive visits by the server

Let $V_i(t)$ denote the expected number of restrictive visits by the server in (0,t], given that the system entered the regenerative state i at t=0, for model-II.
The recursive relations of $V_i(t)$ are given as

$V_i(t) = \sum_{ij} Q_{ij}(t) \boxed{s} [\delta_i + V_j(t)],\ s.t.$

$\delta_i = \begin{cases} 1; if\ there\ is\ a\ visit\ in\ transition\ from\ i \to j \\ 0; otherwise \end{cases}$ (10.1)

Solving above differential- difference equations (10.1) for $\tilde{V}_0(s)$ using Laplace and Laplace- Stiltje's transforms, and letting t→∞ i.e. s→0 we get, asymptotically, the expected number of visits per unit time of the server as

$V_{20}(\infty) = V_{20} = \lim_{s\to 0} s\tilde{N}_0(s) = \frac{N_{25}}{D_{22}}$ (10.2)

$N_{25} = \{b + ah^*(2\lambda)g^*(2\lambda)\}\{1 - w^*(2\lambda)\}$ (10.3)

11. Cost Analysis

11.1 The expected profit gained in (0, t]

Profit= total revenue in (0,t]- total expenditure incurred in (0, t] i.e.

$P_i(t) = [k_1\mu_{up}(t)] - [k_2\mu_b(t) + k_3N_{i0}(t) + k_4V_{20}(t)]$ (11.1.1)

Where k_1= Revenue per unit up time for the system
k_2= Cost per unit time for which the system is under repair
k_3= Cost per visit by the server
k_4= Cost per restrictive visit by the server in model-II

11.2 Expected profit per unit time in steady state is given by

$P_i(\infty) = \lim_{t\to\infty} \frac{P_i(t)}{t} = \lim_{s\to 0} s^2 P_i^*(s)$ for i=1,2 (11.2.1)

$P_i = k_1A_{i0} - k_2B_{i0} - k_3N_{i0} - k_4V_{i0}$ (Note: for i=1, $V_{10} = 0$) (11.2.2)

Summary

A 2(k)-out-of-3(n) cold standby system of identical units with arbitrary distribution of repair and inspection under operational restrictions is studied in this paper. Expressions for various system performance characteristics are drawn by using semi-Markov processes and re-generative point technique. By using these expressions, the analytical as well numerical solutions of measures of performance can be obtained for some specific systems in transient and steady states. The models developed in this paper are sufficiently applicable, with corresponding minor or major modifications, to many industrial or real systems such as power plant, communication system, and waste-water treatment plant etc.

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