Journal of Reliability and Statistical Studies [ISSN: 0974-8024 (Print), 2229-5666 (Online)]
Vol. 3, Issue 2 (2010):27-34

POSTERIOR ESTIMATES OF TWO PARAMETER EXPONENTIAL DISTRIBUTION USING S-PLUS SOFTWARE

S.P. Ahmad[1] and Bilal Ahmad Bhat[2]
1. Department of Statistics, University of Kashmir, Srinagar (India)
E Mail: sprvz@yahoo.com
2. Division of Sericulture, SKUAST-K, Shalimar, Srinagar (India)

Abstract

This paper deals with the Bayesian estimation of parameters of exponential distribution under different priors. The numerical and graphical illustration of posterior densities of the parameters of interest has been done in S-PLUS Software.

Keywords: Exponential distribution, Marginal posterior densities, Posterior estimates and S-PLUS Software.

1. Introduction

Exponential distribution is a widely used lifetime distribution which has appeared in the literature since the early 1800s. This distribution is one of the commonly used statistical distributions in practice. Sukhatme (1937), Epstein and Sobel (1953, 1954, 1955), Epstein (1954), Bartholomew (1957), Mendenhall (1958), Johnson, Kotz and Balakrishnan (1994, 1995), Lawless (2003) and others have discussed this distribution with applications.

As given in Sinha 1986, the pdf of two parameter exponential distribution is given by

$$f(y)=\frac{1}{\theta}e^{-\frac{1}{\theta}(y-\mu)} \quad , \qquad -\infty<\mu<y<\infty, \quad \theta>0 \tag{1}$$

If $y_1, y_2, ..., y_n$ be iid observations from an exponential distribution, then the likelihood function is given by

$$L(\mu,\theta)=\frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}(y_i-\mu)}=\frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}\{(y_i-y_{(1)})+(y_{(1)}-\mu)\}}$$

$$=\frac{1}{\theta^n}e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}} \tag{2}$$

Where $y_{(1)}$ is the first order statistic in the sample $y=(y_1, y_2, ..., y_n)$ such that $y_{(1)} \le y_{(2)} \le ... \le y_{(n)}$ and $S=\sum_{i=1}^{n}(y_i-y_{(1)})$.

Jeffrey (1961) and others make extensive use of improper prior pdf's. Let us consider a more general class of priors,

$$p(\mu,\theta)\ \alpha\ \left(\frac{1}{\theta^{c}}\right) \qquad c \geq 0 \tag{3}$$

2. Posterior density for μ and θ

According to Bayes theorem, we have Posterior density α (prior density *likelihood) i.e. $p(\mu,\theta \mid y)\ \alpha\ p(\mu,\theta)\,L(\mu,\theta)$

from the equations (2) and (3), the posterior density of μ and θ is given by

$$p(\mu,\theta \mid y) \quad \alpha \quad \frac{1}{\theta^{n+c}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}}$$

$$= \frac{K}{\theta^{n+c}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}}$$

Where K is a normalizing constant and is given by

$$K^{-1} = \int_{-\infty}^{y_{(1)}} \int_{0}^{\infty} \frac{1}{\theta^{n+c}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}} d\mu\, d\theta$$

$$= \frac{\Gamma(n+c-2)}{n\, S^{n+c-2}}$$

$$\Rightarrow \qquad K = \frac{n\, S^{n+c-2}}{\Gamma(n+c-2)}$$

$$\therefore \qquad p(\mu,\theta \mid y) = \frac{n \quad S^{n+c-2}}{\Gamma(n+c-2) \quad \theta^{n+c}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}} \tag{4}$$

For $c = 0,\ p(\theta) = 1$ (uniform prior), (4) becomes

$$p(\mu,\theta \mid y) = \frac{n \quad S^{n-2}}{\Gamma(n-2) \quad \theta^{n}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}}$$

For $c = 1,\ p(\theta) = \frac{1}{\theta}$, (4) becomes

$$p(\mu,\theta \mid y) = \frac{n \quad S^{n-1}}{\Gamma(n-1) \quad \theta^{n+1}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}}$$

For $c = 2,\ p(\theta) = \frac{1}{\theta^{2}}$, (4) becomes

$$p(\mu,\theta \mid y) = \frac{n \quad S^{n}}{\Gamma(n) \quad \theta^{n+2}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}}$$

3. Marginal Posterior densities for μ and θ

The marginal posterior density of μ is given by

$$p(\mu \mid y) = \int_0^\infty p(\mu,\theta \mid y)\, d\theta$$

$$= \frac{n\, S^{n+c-2}}{\Gamma(n+c-2)} \int_0^\infty \frac{1}{\theta^{n+c}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}} d\theta$$

$$= \frac{n\, S^{n+c-2}}{\Gamma(n+c-2)} \frac{\Gamma(n+c-1)}{\{S+n(y_{(1)}-\mu)\}^{(n+c-1)}}$$

$$= n\,(n+c-2) \frac{S^{n+c-2}}{\{S+n(y_{(1)}-\mu)\}^{n+c-1}} \qquad (5)$$

For $c=0,\ p(\theta)=1$, (5) becomes $p(\mu \mid y) = n\,(n-2) \dfrac{S^{n-2}}{\{S+n(y_{(1)}-\mu)\}^{n-1}}$

For $c=1,\ p(\theta)=\dfrac{1}{\theta}$, (5) becomes $p(\mu \mid y) = n\,(n-1) \dfrac{S^{n-1}}{\{S+n(y_{(1)}-\mu)\}^{n}}$

For $c=2,\ p(\theta)=\dfrac{1}{\theta^2}$, (5) becomes $p(\mu \mid y) = n^2 \dfrac{S^{n}}{\{S+n(y_{(1)}-\mu)\}^{n+1}}$

The marginal posterior density of θ is given by

$$p(\theta \mid y) = \int_{-\infty}^{y_{(1)}} p(\mu,\theta \mid y)\, d\mu$$

$$= \frac{n\, S^{n+c-2}}{\Gamma(n+c-2)} \frac{1}{\theta^{n+c}} \int_{-\infty}^{y_{(1)}} e^{-\frac{1}{\theta}\{S+n(y_{(1)}-\mu)\}} d\mu$$

$$= \frac{n\, S^{n+c-2}}{\Gamma(n+c-2)} \frac{1}{\theta^{n+c}} \frac{\theta\, e^{-\frac{S}{\theta}}}{n}$$

$$= \frac{S^{n+c-2}}{\Gamma(n+c-2)} \frac{e^{-\frac{S}{\theta}}}{\theta^{n+c-1}} \qquad (6)$$

For $c=0,\ p(\theta)=1$, (6) becomes $p(\theta \mid y) = \dfrac{S^{n-2}}{\Gamma(n-2)} \dfrac{e^{-\frac{S}{\theta}}}{\theta^{n-1}}$

For $c=1,\ p(\theta)=\dfrac{1}{\theta}$, (5) becomes $p(\theta \mid y) = \dfrac{S^{n-1}}{\Gamma(n-1)} \dfrac{e^{-\frac{S}{\theta}}}{\theta^{n}}$

For $c = 2,\ p(\theta) = \frac{1}{\theta^2}$, (5) becomes $p(\theta \mid y) = \frac{S^n}{\Gamma(n)} \frac{e^{-\frac{S}{\theta}}}{\theta^{n+1}}$

4. Posterior estimates of μ and θ

The posterior estimate of μ is given by

$$E(\mu \mid y) = \int_{-\infty}^{y_{(1)}} \mu \ p(\mu \mid y) \ d\mu$$

$$= n\ (n+c-2)\ S^{n+c-2} \int_{-\infty}^{y_{(1)}} \frac{\mu}{\{S + n(y_{(1)} - \mu)\}^{n+c-1}} d\mu$$

$$= y_{(1)} - \frac{S}{n(n+c-3)} \qquad (7)$$

For $c = 0,\ p(\theta) = 1$, (7) becomes $E(\mu \mid y) = y_{(1)} - \frac{S}{n(n-3)}$

For $c = 1,\ p(\theta) = \frac{1}{\theta}$, (7) becomes $E(\mu \mid y) = y_{(1)} - \frac{S}{n(n-2)}$

For $c = 2,\ p(\theta) = \frac{1}{\theta^2}$, (7) becomes $E(\mu \mid y) = y_{(1)} - \frac{S}{n(n-1)}$

The Posterior estimates of θ is given by

$$E(\theta \mid y) = \int_0^{\infty} \theta \ p(\theta \mid y) \ d\theta$$

$$= \frac{S^{n+c-2}}{\Gamma(n+c-2)} \int_0^{\infty} \frac{e^{-\frac{S}{\theta}}}{\theta^{n+c-2}} d\theta$$

$$= \frac{S}{n+c-3} \qquad (8)$$

For $c = 0,\ p(\theta) = 1$, (8) becomes $E(\theta \mid y) = \frac{S}{(n-3)}$

For $c = 1,\ p(\theta) = \frac{1}{\theta}$, (8) becomes $E(\theta \mid y) = \frac{S}{(n-2)}$

For $c = 2,\ p(\theta) = \frac{1}{\theta^2}$, (8) becomes $E(\theta \mid y) = \frac{S}{(n-1)}$

5. Numerical & Graphical Illustration (Grubbs, F.E., 1971)

Nineteen military personnel carriers failed in services for one reason or the other at the following mileages: 162, 200, 271, 302, 393, 508, 539, 629, 706, 777, 884, 1008, 1101, 1182, 1463, 1603, 1984, 2355 and 2880 miles. Numerical and graphical illustrations are implemented in S-PLUS Software for two parameter exponential distribution. Posterior estimates of μ and θ are given in Table I. The graphical representation for marginal posterior densities of μ and θ are shown in Figures 1 and 2 respectively. Moreover, we have developed the function for estimating parameters μ and θ of two parameter exponential distribution under different priors & is given in Appendix A. Also, functions for graphical representation of the marginal densities of μ and θ under different priors were also developed in S-PLUS and are given in Appendix B & C respectively.

Prior	**Posterior mean of μ**	**Posterior mean of θ**
1	109.7993	991.8125
$\frac{1}{\theta}$	112.8700	933.4706
$\frac{1}{\theta^2}$	115.5994	881.6111

Table I: Posterior estimates of μ and θ under different priors using S-PLUS

The posteriors of μ and θ are plotted in figures 1 & 2 respectively. The posteriors μ are quite robust for varying c in the prior $p(\mu,\theta)\ \alpha\ \left(\frac{1}{\theta^c}\right)$ while the posteriors of θ is less robust.

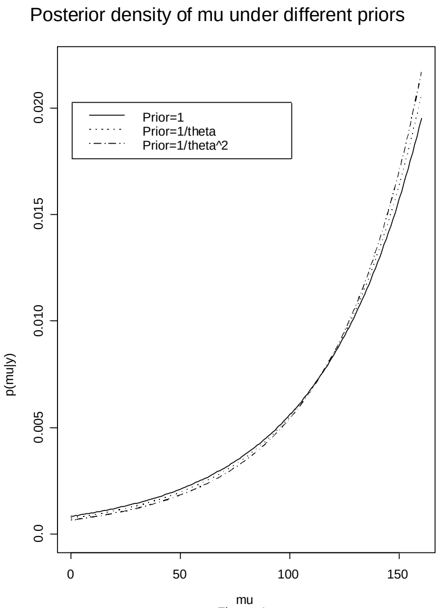

Figure: 1

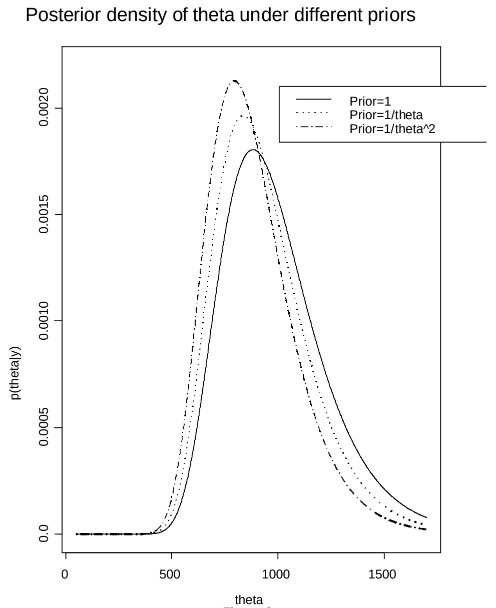

Figure: 2

Appendix A: Function for estimating parameters μ and θ of two parameter exponential distribution.

```
Mu.theta<-function(y)
 {
 n<-length(y)
 C<-c(0,1,2)
 y1<-min(y)
 s<-sum(y-y1)
 estimate1<-y1-(s/(n*(n+C-3)))
 estimate2<-s/(n+C-3)
 list(mu=estimate1,theta=estimate2)
  }

> y<-c(162, 200, 271, 302, 393, 508, 539, 629, 706, 777,
884, 1008, 1101, 1182, 1463, 1603, 1984, 2355,2880 )

>  Mu.theta(y)
```

Appendix B: Function for graphical representation of the marginal density of μ under different priors.

```
mu.plot<-function(y)
{
n<-length(y)
y1<-min(y)
s<-sum(y-y1)
mu<-seq(0,160)
pmu<-(n*(n-2))*(s^(n-2))/((s+n*(y1-mu))^(n-1))
plot(mu,pmu,xlab="mu",ylab="p(mu|y)",ylim=c(0,0.022),
     main= "Posterior density of mu under different priors",
     sub="Figure: 1",type="l",lty=1)
pmu1<-(n*(n-1))*(s^(n-1))/((s+n*(y1-mu))^(n))
lines(mu,pmu1,lty=2)
pmu2<-(n*(n))*(s^(n))/((s+n*(y1-mu))^(n+1))
lines(mu,pmu2,lty=3)
}
> y<-c(162, 200, 271, 302, 393, 508, 539, 629, 706, 777, 884,
1008, 1101,1182, 1463, 1603, 1984, 2355,2880 )
> mu.plot(y)
> leg.names<c("Prior=1","Prior=1/theta","Prior=1/theta^2")
> legend(locator(1),leg.names,lty=1:3)
```

Appendix C: Function for graphical representation of the marginal density of θ under different priors.

```
theta.plot<-function(y)
{
n<-length(y)
y1<-min(y)
s<-sum(y-y1)
theta<-seq(50,1700)
ptheta<- (s^(n-2))*(exp(-s/theta))/((gamma(n-2))*(theta^(n-1)))
ptheta1<- (s^(n-1))*(exp(-s/theta))/((gamma(n-1))*(theta^(n)))
ptheta2<- (s^n)*(exp(-s/theta))/((gamma(n))*(theta^(n+1)))
plot(theta,ptheta,xlab="theta",ylab="p(theta|y)",ylim=c(0,.0022),
     main= "Posterior density of theta under different priors",
     sub="Figure: 2",type="l",lty=1)
lines(theta,ptheta1,lty=2)
lines(theta,ptheta2,lty=3)
}
> y<-c(162, 200, 271, 302, 393, 508, 539, 629, 706, 777, 884,
1008, 1101,1182, 1463, 1603, 1984, 2355,2880 )
> theta.plot(y)
> leg.names<-c("Prior=1","Prior=1/theta","Prior=1/theta^2")
> legend(locator(1),leg.names,lty=1:3)
```

References

1. Bartholomew, D.J. (1957). A problem in life testing, Journal of American Statistical Association, 52, p. 350-355.
2. Epstein, B. (1954). Truncated life testing in exponential case, Annals of the Mathematical Statistics, 25, p. 555-564.
3. Epstein, B. and Sobel, M. (1953). Life testing, Journal of the American Statistical Association, 48, p. 486-502.
4. Epstein, B. and Sobel, M. (1954). Some theorems relevant to life testing from an exponential distribution, Annals of the Mathematical Statistics, 25, p. 373-381.
5. Epstein, B. and Sobel, M. (1955). Sequential life testing in exponential case, Annals of the Mathematical Statistics, 26, p. 82-93.
6. Grubbs, F.E. (1971). Approximate fiducial bounds on reliability for the two parameter negative exponential distribution, Technometrics, Vol. 13, No. 4, p. 873-876.
7. Jeffrey, H. (1961). Theory of Probability, Oxford University Press.
8. Johnson, N.L., Kotz, S. and Balakrishnan, N. (1994). Continuous Univariate Distributions, Vol. 1, John Wiley & Sons, New York.
9. Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). Continuous Univariate Distributions, Vol. 2, John Wiley & Sons, New York.
10. Johnson, R.A. (1970). Asymptotic expansions associated with posterior distributions, Annals of Mathematical Statistics, 41, p. 196-215.
11. Lawless, J.F. (2003). Statistical Models and Methods for Lifetime Data. Wiley, New York.
12. Mendenhall, W. (1958). A bibliography on life testing and related topics. Biometrika 45, 521-543.
13. Sinha, S.K. (1986). Reliability and Life Testing. Wiley Eastern Limited.