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RELIABILITY AND HAZARD RATE ESTIMATION OF A LIFE TESTING MODEL

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Abstract

The present paper deals with the reliability and hazard rate estimation of a Weibull type life testing model. Its use as a life testing model has also been illustrated. The proposed model has been found better then exponential for several sets of lifetime data. Some characteristics of the model have also been investigated.

Key Words: Reliability Function, Hazard Rate Function, Life Testing Model, Maximum likelihood estimator, Moment generating function, Failure censoring.

1. Introduction

A number of life testing models have been used in the literature for the real life situations. Among which, the most commonly used model for most of the real life situations is the well known exponential model. It has constant hazard rate. For some situations, where hazard rate increases polynomially, we use Weibull distribution. The Weibull distribution is commonly used model to failure time data, since it generalizes the exponential distribution allowing for the power dependence of the hazard rate function on time. This power dependence is controlled by the distribution shape parameter. It has increasing failure rate when the shape parameter is greater than one and has decreasing failure rate when shape parameter is smaller than one. It is thus important to reliably draw inferences on such a parameter. It is also felt that many real life situations may demand a model for which hazard rate increases non polynomially, the proposed model is an effort in this direction. In this model, α (shape parameter) is supposed to be positive rational number rather than positive integer. The probability density function of the proposed model is given by

$$f(x) = \frac{\alpha}{\beta^{\alpha}} e^{-(x/\beta)^{\alpha}} x^{\alpha-1} I_{(0,\infty)}(x) \tag{1}$$

where α and $\beta>0$ are parameters and X is a random variable.

The proposed model seems to be fit in many real life time models for different values of α. It is investigated that for many sets of lifetime data published in Davis (1952), it is a better fit than exponential model for $\alpha=4/3$.

2. Characteristics of the proposed model

2.1. Moment generating function

M.G.F. of the random variable X is defined as

$$M_X(t) = E\left[e^{tx}\right] = \int_x e^{xt} f(x)dx$$

$$= \int_0^\infty e^{tx} . \frac{\alpha}{\beta^\alpha} e^{-(x/\beta)^\alpha} x^{\alpha-1} dx$$

$$= \int_0^\infty e^{t\beta z^{1/\alpha}} e^{-z} dz \qquad \text{put } \left\{ x/\beta \right\}^\alpha = z$$

$$= \sum_{r=0}^\infty \frac{t^r \beta^r}{r!} \int_0^\infty e^{-z} z^{r/\alpha} dz$$

$$= \sum_{r=0}^\infty \frac{t^r \beta^r}{r!} \Gamma\left(\frac{r}{\alpha} + 1\right) \tag{2}$$

Now, μ'_s=coefficient of $t^s/s!$ in the expansion of $M_x(t)$

$$\mu'_s = \beta^s \Gamma\left(\frac{s}{\alpha} + 1\right)$$

$$\mu'_s = \beta^s \Gamma\left(\frac{3s}{4} + 1\right) \quad \text{for } \alpha = 4/3 \tag{3}$$

$$\mu'_1 = \frac{3}{4} \beta \Gamma\left(3/4\right) = \text{Mean}$$

$$\mu'_2 = \frac{3}{4} \beta^2 \Gamma\left(1/2\right)$$

$$\mu'_3 = \frac{45}{64} \beta^3 \Gamma\left(1/4\right)$$

$$\mu'_4 = 6\beta^4 \tag{4}$$

2.2 Central Moments

$$\mu_2 = \frac{3}{4} \beta^2 [\Gamma\left(1/2\right) - \frac{3}{4} \{\Gamma\left(3/4\right)\}^2] \tag{5}$$

$$\mu_3 = \frac{9}{64} \beta^3 [5\Gamma\left(1/4\right) - 12\Gamma\left(1/2\right)\Gamma\left(3/4\right) + 6\{\Gamma\left(3/4\right)\}^3] \tag{6}$$

$$\mu_4 = \frac{3}{256} \beta^4 [512 - 180\Gamma\left(1/4\right)\Gamma\left(3/4\right) + 216\Gamma\left(1/2\right)\{\Gamma\left(3/4\right)\}^2 - 81\{\Gamma\left(3/4\right)\}^4] \tag{7}$$

2.3 Coefficients of Skewness and Kurtosis

The coefficient of skewness (g_1) is given by

$$\gamma_1 = \frac{\sqrt{3}[5\Gamma\left(1/4\right) - 12\Gamma\left(1/2\right)\Gamma\left(3/4\right) + 6\{\Gamma\left(3/4\right)\}^3]}{8[\Gamma\left(1/2\right) - \frac{3}{4}\{\Gamma\left(3/4\right)\}^2]^2} \times 0.354 \quad (8)$$

$$\gamma_2 = \frac{[512 - 180\Gamma\left(1/4\right)\Gamma\left(3/4\right) + 216\Gamma\left(1/2\right)\{\Gamma\left(3/4\right)\}^2 - 81\{\Gamma\left(3/4\right)\}^4]}{48[\Gamma\left(1/2\right) - \frac{3}{4}\{\Gamma\left(3/4\right)\}^2]^2} \times 2 \quad (9)$$

Here, $\gamma_1 > 0$ and $\gamma_2 > 0$, hence the proposed model is positively skewed and leptokurtic for $\alpha = 4/3$.

2.4 Reliability and Hazard Rate Function

Reliability function is defined as

$R(t) = P[X>t] = 1- P[X\leq t]$

$$= 1- F_x(t) = e^{-(t/\beta)^{\alpha}} \quad (10)$$

which is a non-increasing function of t satisfying $R(0)=1$ and $\lim_{t\to\infty} R(t) = 0$.
The hazard rate function can be defined as

$$H(t) = f(t)/R(t) = \frac{\alpha t^{\alpha-1}}{\beta^{\alpha}} \quad (11)$$

which is a non-decreasing function of t.

3. Estimation of Parameters

3.1. Maximum likelihood estimator of β [α is known]

(i) When sample is uncensored

Suppose n items are subjected to test and the test is terminated after all the items have failed. Let $X_1, X_2, \ldots, X_n$ be the random failure times and suppose $X_1, X_2, \ldots, X_n$ is a random sample from the proposed model with p.d.f. (1). Then the likelihood function is given by

$$L = \frac{\alpha^n}{\beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right) e^{-\sum_{i=1}^{n}(x_i/\beta)^{\alpha}} \prod_{i=1}^{n} I_{(0,\infty)}^{(x_i)} \quad (12)$$

Taking logarithms both sides and differentiating log L partially with respect to β and equating it to zero, we get,

$$\beta^{\alpha} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}$$

$$\beta^{*} = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right]^{1/\alpha}$$

Thus, the maximum likelihood estimator of β is the α^{th} root of the arithmetic mean of α power of observations.

$$\beta^* = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{4/3}\right]^{3/4} \quad for \ \alpha = \frac{4}{3} \tag{13}$$

(ii) When the sample is censored

Let us consider the sample as failure censored i.e. if we put n items on test and may terminate the experiment when a pre-assigned number say (r<n) of items have failed. The samples obtained from such an experiment are called failure censored samples. Let the data consist of the failure times of r items (say $X_1, X_2, \ldots\ldots, X_r$) and the fact that (n-r) items have survived beyond $X_{(r)}$. In this situation, r , the number of items that have failed is fixed while $X_{(r)}$, the time at which the experiment is terminated is a random variable. Gross and Clark (1975) suggested a technique for obtaining the MLE in case of random sampling. According to him, the likelihood function is

$$L = \prod_{i=1}^{n}\left[f(x_i)\right]^{\delta_i}\left[R(x_i)\right]^{1-\delta_i} \tag{14}$$

Where δ_i=1, if i^th item fails

= 0, otherwise

For the proposed model

$$L = \prod_{i=1}^{n}\left[\frac{\alpha x_i^{\alpha-1}}{\beta^{\alpha}} e^{-(x_i/\beta)^{\alpha}} I_{(0,\infty)}^{(x_i)}\right]^{\delta_i}\left[e^{-(x_i/\beta)^{\alpha}}\right]^{1-\delta_i} \tag{15}$$

Keeping in view the fact that (n-r) items are survived for x_r time atleast, the likelihood function may be written as

$$L = \frac{\alpha^r}{\beta^{r\alpha}}\prod_{i=i}^{r} x_i^{\alpha-1} e^{-\sum_{i=1}^{r}(x_i/\beta)^{\alpha}}\left[e^{-(x_r/\beta)^{\alpha}}\right]^{n-r}$$

$$L = \frac{\alpha^r}{\beta^{r\alpha}}\prod_{i=i}^{r} x_i^{\alpha-1} e^{-\beta^{-\alpha}\left[\sum_{i=1}^{r} x_i^{\alpha} + (n-r)x_i^{\alpha}\right]}$$

Taking logarithm on both sides, differentiate with respect to β and equating it to zero, we get

$$\beta^* = \left[\frac{1}{r}\left\{\sum_{i=1}^{r} x_i^{\alpha} + (n-r)x_r^{\alpha}\right\}\right]^{1/\alpha}$$

Which is the MLE of β for failure censored sample.

and

$$\beta^* = \left[\frac{1}{r}\sum_{i=1}^{r} x_i^{4/3} + \frac{1}{r}(n-r)x_r^{4/3}\right]^{3/4} \quad for \ \alpha = \frac{4}{3} \tag{16}$$

3.2 Maximum Likelihood estimates of Reliability and Hazard rate function

Reliability and hazard rate functions are nothing but function of parameters of the model. The maximum likelihood estimators of reliability and hazard rate function are dented by R*(t) and H*(t) respectively.

(i) When sample is uncensored

$$R^*(t) = 1 - F^*(t)$$

$$R^*(t) = e^{-nt^{\alpha}/\sum_{i=1}^{n} x_i^{\alpha}}$$

$$H^*(t) = \frac{n \propto t^{\propto-1}}{\sum_{i=1}^{n} x_i^{\propto}}$$

For α = 4/3

$$R^*(t) = e^{-nt^{4/3}/\sum_{i=1}^{n} x_i^{4/3}} \quad (17)$$

$$H^*(t) = \frac{4}{3}\frac{nt^{1/3}}{\sum_{i=1}^{n} x_i^{4/3}} \quad (18)$$

(ii) When the sample is censored

$$R^*(t) = e^{-rt^{\propto}\left[\sum_{i=1}^{r} x_i^{\propto}+(n-r)x_r^{\propto}\right]^{-1}}$$

$$H^*(t) = r \propto t^{\propto-1}\left[\sum_{i=1}^{r} x_i^{\propto} + (n-r)x_r^{\propto}\right]^{-1}$$

For α = 4/3

$$R^*(t) = e^{-rt^{4/3}\left[\sum_{i=1}^{r} x_i^{4/3}+(n-r)x_r^{4/3}\right]^{-1}} \quad (19)$$

$$H^*(t) = \frac{4}{3} rt^{1/3}\left[\sum_{i=1}^{r} x_i^{4/3} + (n-r)x_r^{4/3}\right]^{-1} \quad (20)$$

4. Applications of the proposed model

The proposed model can be used in the analysis of some life failure data related to real life situations. Davis (1952) summarizes the rationale and statistical techniques employed in the analysis of some failure data obtained from operations performed by machine and people. These data are compared to frequency distributions arising from exponential or normal theory of failure. We have used some sets of data due to Davis (1952) and found that the proposed model is better than the exponential model for many sets of data since the calculated value of chi-square is less than that of its value in case of exponential model. For three sets of data due to Davis (1952), the proposed model has been fitted and theoretical frequencies have been obtained and have been compared with the exponential model. The theoretical frequencies obtained by fitting the proposed model seem to be quite closer to the observed frequencies in comparison to the theoretical frequencies obtained by fitting exponential model. These frequencies as well as the values of reliability and hazard rate functions at different points of time are given in the following tables.

For α = 4/3, $\beta^* = 38.4875$ $\beta^*_{exp} = 35.1764$

Distance	Observed Fre.	Expected Fre. (Proposed)	Expected Fre. (Exponential)
0-20	29	29.03	36.86
20-40	27	26.31	20.88
40-60	14	15.72	11.82
60-80	8	7.95	6.70
80-	7	5.99	8.74
Total	85	85.00	85.00

Table 1: Calculation of theoretical frequencies for data related to "Fifth bus motor failure"

χ^2 =0.38 (Proposed) $\chi^2_{(3, 0.05)} = 7.815$

χ^2 =4.44 (Exponential) $\chi^2_{(3, 0.01)} = 11.341$

Distance	Reliability	Hazard Rate
0	1.0000	0.0000
20	0.6585	0.0279
40	0.3490	0.0351
60	0.1640	0.0402
80	0.0705	0.0442
100	0.0281	0.0476

Table 2: Calculation of Reliability and Hazard rates for data related to "Fifth bus motor failure"

For α = 4/3, β^* = 57.5229 β^*_{exp} = 52.5742

Distance	Observed Fre.	Expected Fre. (Proposed)	Expected Fre. (Exponential)
0-20	27	21.91	31.96
20-40	16	24.55	21.85
40-60	18	19.48	14.93
60-80	13	13.68	10.21
80-100	11	8.90	6.98
100-	16	12.48	15.07
Total	101	101.00	101.00

Table 3: Calculation of theoretical frequencies for data related to "Third bus motor failure"

χ^2 =5.74 (Proposed) $\chi^2_{(4, 0.05)} = 9.488$

χ^2 =6.102 (Exponential) $\chi^2_{(4, 0.01)} = 13.277$

Distance	Reliability	Hazard Rate
0	1.0000	0.0000
20	0.7016	0.0236
40	0.4094	0.0298
60	0.2158	0.0341
80	0.1053	0.0375
100	0.0483	0.0404
120	0.0210	0.0429

Table 4: Calculation of Reliability and Hazard rates for data related to "Third bus motor failure"

For $\alpha = 4/3$, $\beta^* = 161.4720$ $\beta^*_{exp} = 147.5806$

Distance	Observed Fre.	Expected Fre. (Proposed)	Expected Fre. (Exponential)
0-50	22	17.58	26.73
50-100	16	20.57	20.50
100-150	15	17.29	12.11
150-200	12	12.98	9.67
200-300	18	15.12	11.81
300-	10	9.46	12.18
Total	93	93.00	93.00

Table 5: Calculation of theoretical frequencies for data related to "R348 resister used in receiver"

$\chi^2 = 3.082$ (Proposed) $\chi^2_{(4, 0.05)} = 9.488$

$\chi^2 = 6.714$ (Exponential) $\chi^2_{(4, 0.01)} = 13.277$

Distance	Reliability	Hazard Rate
0	1.0000	0.0000
50	0.8110	0.0056
100	0.5899	0.070
150	0.4040	0.0081
200	0.2644	0.0089
250	0.1668	0.0096
300	0.1019	0.0105

Table 6: Calculation of Reliability and Hazard rates for data related to "R348 resister used in receiver"

5. Conclusion

The proposed model has been found as a better fit for many sets of data due to Davis (1952) than the well known exponential model when $\alpha=4/3$, as shown in the abovementioned Tables. For different values of α, this may cover a wide range of life time data.

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