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ESTIMATION OF THE RELIABILITY FUNCTION FOR A FAMILY OF LIFETIME DISTRIBUTIONS UNDER TYPE I AND TYPE II CENSORINGS

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Abstract

A family of lifetime distributions is proposed. The problems of estimating the reliability function R(t)=P(X>t) and P=P(X>Y) are considered under type I and II censorings. Uniformly minimum variance unbiased and maximum likelihood estimators are derived. A comparative study of the performance of the two methods of estimation is done. Simulation study is performed.

Key Words: Reliability function; stress-strength set-up; type I and II censorings; uniformly minimum variance unbiased estimators; maximum likelihood estimators; bootstrap methods.

1. Introduction

The reliability function R(t) is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item, then R(t)=P(X>t). Another measure of reliability under stress-strength set-up is the probability P=P(X>Y), which represents the reliability of an item of random strength X subject to random stress Y. Many researchers have considered the problems of estimation of R(t) and 'P' in the literature under censoring and complete sample case. Uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES) for R(t) and 'P' have been derived for various lifetime distributions, like, exponential, gamma, Weibull, half-normal, Maxwell, Rayleigh, Burr and others. For a brief review, one may refer to Pugh (1963), Basu (1964), Bartholomew (1957, 1963), Tong (1974, 1975), Johnson (1975), Chaturvedi and Surinder (1999), Sinha (1986), Kelly, Kelly and Schucany (1976), Chao (1982), Sathe and Shah (1981), Constantine, Karson and Tse (1986), Tyagi and Bhattacharya (1989), Awad and Gharraf (1986) and others.

In the present paper, we consider a family of distributions, which covers many lifetime distributions as specific cases. The UMVUES and MLES of R(t) and 'P' are derived under type I and II censorings. In order to obtain these estimators, the major role is played by the estimators of the powers of the parameter(s) and the functional forms of the parametric functions to be estimated are not needed. It is worth mentioning here that, in order to estimate 'P', in the literature, the authors have considered the cases when X and Y follow the same distributions, may be with different parameters. We have generalized the result to the case when X and Y may follow any distribution from the proposed family of distributions. Simulation study is carried out to investigate the performance of the estimators.

In Section 2, we introduce the family of lifetime distributions. In Section 3, we derive the UMVUES of R(t) and ‘P’ under type I and II censorings. In Section 4, we obtain the MLES of R(t) and ‘P’ under two types of censorings. Finally, in Section 5, the simulation study is performed.

2. The Family of Lifetime Distributions

Let the rv X follow the distribution having the probability density function (pdf)

$$f(x;a,\lambda,\underline{\theta})=\lambda G'(x;a,\underline{\theta})\exp\{-\lambda G(x;a,\underline{\theta})\};\ x>a\geq 0,\ \lambda>0\ . \qquad (2.1)$$

Here, $G(x;a,\underline{\theta})$ is a function of x and may also depend on the parameters ‘a’ and $\underline{\theta}$ - may be vector-valued. Moreover, $G(x;a,\underline{\theta})$ is monotonically increasing in x with $G(a;a,\underline{\theta})=0$, $G(\infty;a,\underline{\theta})=\infty$ and $G'(x;a,\underline{\theta})$ denotes the derivative of $G(x;a,\underline{\theta})$ with respect to x.

We note that (2.1) represents a family of lifetime distributions as it covers the following lifetime distributions as specific cases:

i. For $G(x;a,\underline{\theta})=x$ and a=0, we get the one-parameter exponential distribution [Johnson and Kotz (1970, p.166)].

ii. For $G(x;a,\underline{\theta})=x^p$ (p>0) and a=0, it gives Weibull distribution [Johnson and Kotz (1970, p.250)].

iii. For $G(x;a,\underline{\theta})=x^2$ and a=0, it leads us to Rayleigh distribution [Sinha (1986, p.200)].

iv. For $G(x;a,\underline{\theta})=\log(1+x^b)$ (b>0) and a=0, it turns out to be Burr distribution [Burr (1942) and Cislak and Burr (1968)].

v. For $G(x;a,\underline{\theta})=\log(\frac{x}{a})$, it is known as Pareto distribution [Johnson and Kotz (1970, p.233)].

vi. For $G(x;a,\underline{\theta})=\log(1+\frac{x}{v})$, v>0 and a=0, we get Lomax (1954) distribution.

vii. For $G(x;a,\underline{\theta})=\log(1+\frac{x^b}{v})$, b>0, v>0 and a=0, it gives Burr distribution with scale parameter v [see Tadikamalla (1980)].

viii. For $G(x;a,\underline{\theta})=x^{\gamma}\exp(\nu x)$, $\gamma>0$, $\nu>0$ and a=0, it leads us to the modified Weibull distribution of Lai *et al.* (2003).

ix. For $G(x;a,\underline{\theta})=\gamma\exp(\frac{x^{\nu}}{\gamma^{\nu}}-1)$, $\gamma>0$, $\nu>0$ and a=0, it turns out to be a modified form of Weibull distribution considered by Xie *et al*. (2002). If we also take $\gamma=1$, this reduces to the lifetime distribution considered by Chen (2000).

x. For $G(x;a,\underline{\theta})=(x-a)+\frac{\nu}{\lambda}\log(\frac{x+\nu}{a+\lambda}),\nu>0,\lambda>0$, we get the generalized Pareto distribution of Ljubo (1965).

3. UMVUE's of R(t) and ‘P’ under Type I and Type II Censorings

Throughout this section, we assume that λ is unknown, but ‘a’ and $\underline{\theta}$ are known. First we consider the estimation based on type II censored data. Suppose n items are put on a test and the test is terminated after the first r ordered observations are recorded. Let $a \le X_{(1)} \le X_{(2)} \le \ldots \le X_{(r)}$, $0<r<n$, be the lifetimes of first r ordered observations. Obviously, (n-r) items survived until $X_{(r)}$.

Lemma 1: Let $S_r = \sum_{i=1}^{r} G(X_{(i)};a,\underline{\theta})+(n-r)G(X_{(r)};a,\underline{\theta})$. Then, S_r is complete and sufficient for the family of distributions given at (2.1). Moreover, the pdf of S_r is

$$g(s_r;a,\lambda,\underline{\theta}) = \frac{\lambda^r s_r^{r-1}}{\Gamma(r)} \exp(-\lambda s_r) \tag{3.1}$$

Proof: From (2.1), the joint pdf of $a \le X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ is

$$f^*(x_{(1)},x_{(2)},\ldots,x_{(n)};a,\lambda,\underline{\theta})=n!\,\lambda^n \prod_{i=1}^{n} G'(x_{(i)};a,\underline{\theta}) \exp\{-\lambda \sum_{i=1}^{n} G(x_{(i)};a,\underline{\theta})\}. \tag{3.2}$$

Integrating out $x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n)}$ from (3.2) over the region $x_{(r)} \le x_{(r+1)} \le \ldots \le x_{(n)}$, the joint pdf of $a \le X_{(1)} \le X_{(2)} \le \ldots \le X_{(r)}$ comes out to be

$$h(x_{(1)},x_{(2)},\ldots,x_{(r)};a,\lambda,\underline{\theta})= n(n-1)\ldots(n-r+1)\,\lambda^r \prod_{i=1}^{r} G'(x_{(i)};a,\underline{\theta}) \exp(-\lambda s_r)\,. \tag{3.3}$$

It follows easily from (2.1) that the rv $U=\lambda G(X;a,\underline{\theta})$ follows exponential distribution with mean life $1/\lambda$. Moreover, if we consider the transformation

$$Z_i=(n-i+1)\{U_{(i)}-U_{(i-1)}\},\ i=1,2,\ldots,r;\ U_0=0,$$

then Z_i's are independent and identically distributed (iid) rv's, each having exponential distribution with mean life $1/\lambda$. It is easy to see that $\sum_{i=1}^{r} Z_i = S_r$. Result (3.1) now follows from the additive property of gamma distribution [see Johnson and Kotz (1970, p.170)]. It follows from (3.3) that S_r is sufficient for the family of distributions given at (2.1). Since the distribution of S_r belongs to exponential family of distributions, it is also complete [see Rohatgi (1976, p.347)].

The following lemma provides the UMVUE's of the powers (positive, as well as, negative) of λ.

Lemma 2: For $q \in (-\infty,\infty)$, the UMVUE of λ^q is

$$\hat{\lambda}_{II}^{q} = \begin{cases} \dfrac{\Gamma(r)}{\Gamma(r-q)} S_r^{-q} & (q<r) \\ 0, \text{ otherwise.} \end{cases}$$

Proof: From (3.1),

$$E(S_r^{-q}) = \frac{\lambda^r}{\Gamma(r)} \int_0^\infty s_r^{r-q-1} \exp(-\lambda s_r)\, ds_r$$

$$= \left\{\frac{\Gamma(r-q)}{\Gamma(r)}\right\}\lambda^q$$

and the lemma follows from Lehmann-Scheffé theorem [see Rohatgi (1976, p.357)].

In the following lemma, we provide the UMVUE of the sampled pdf (2.1) at a specified point ‘x’.

Lemma 3: The UMVUE of $f(x;a,\lambda,\underline{\theta})$ at a specified point ‘x’ is

$$\hat{f}_{II}(x;a,\lambda,\underline{\theta})=\begin{cases}\dfrac{(r-1)G'(x;a,\underline{\theta})}{S_r}\left[1-\dfrac{G(x;a,\underline{\theta})}{S_r}\right]^{r-2} & ;G(x;a,\underline{\theta})<S_r\\ 0, \text{ otherwise.}\end{cases}$$

Proof: We can write (2.1) as

$$f(x;a,\lambda,\underline{\theta})=G'(x;a,\underline{\theta})\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}G^i(x;a,\underline{\theta})\lambda^{i+1}. \tag{3.4}$$

Using Lemma 1 of Chaturvedi and Tomer (2002) and Lemma 2, from (3.4), the UMVUE of $f(x;a,\lambda,\underline{\theta})$ at a specified point ‘x’ is

$$\hat{f}_{II}(x;a,\lambda,\underline{\theta}) = G'(x;a,\underline{\theta})\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}G^i(x;a,\underline{\theta})\hat{\lambda}_{II}^{i+1}$$

$$=\frac{(r-1)G'(x;a,\underline{\theta})}{S_r}\sum_{i=0}^{r-2}(-1)^i\binom{r-2}{i}\left\{\frac{G(x;a,\underline{\theta})}{S_r}\right\}^i$$

and the lemma holds.

In the following theorem, we obtain UMVUE of R(t).

Theorem 1: The UMVUE of R(t) is given by

$$\hat{R}_{II}(t)=\begin{cases}\left[1-\dfrac{G(t;a,\underline{\theta})}{S_r}\right]^{r-1}, & G(t;a,\underline{\theta})<S_r\\ 0, \text{ otherwise.}\end{cases}$$

Proof: Let us consider the expected value of the integral $\int_t^{\infty}\hat{f}_{II}(x;a,\lambda,\underline{\theta})\,dx$ with respect to S_r i.e.

$$\int_0^{\infty}\left\{\int_t^{\infty}\hat{f}_{II}(x;a,\lambda,\underline{\theta})\,dx\right\}g(s_r;a,\lambda,\underline{\theta})\,ds_r = \int_t^{\infty}\left[E_{S_r}\left\{\hat{f}_{II}(x;a,\lambda,\underline{\theta})\right\}\right]dx$$

$$=\int_t^{\infty}f(x;a,\lambda,\underline{\theta})\,dx$$

$$= R(t). \tag{3.5}$$

We conclude from (3.5) that the UMVUE of R(t) can be obtained simply integrating $\hat{f}_{II}(x;a,\lambda,\underline{\theta})$ from t to ∞.

Thus, from Lemma 3,

$$\hat{R}_{II}(t)=\frac{(r-1)}{S_r}\int_t^{\infty}G'(x;a,\underline{\theta})\left[1-\frac{G(x;a,\underline{\theta})}{S_r}\right]^{r-2}dx$$

and the theorem follows.

Let X and Y be two independent rv's following the classes of distributions $f_1(x;a_1,\lambda_1,\underline{\theta}_1)$ and $f_2(y;a_2,\lambda_2,\underline{\theta}_2)$, respectively, where

$$f_1(x;a_1,\lambda_1,\underline{\theta}_1)=\lambda_1 G'(x;a_1,\underline{\theta}_1)\exp\{-\lambda_1 G(x;a_1,\underline{\theta}_1)\};\ x>a_1\geq 0,\ \lambda_1>0$$

and

$$f_2(y;a_2,\lambda_2,\underline{\theta}_2)=\lambda_2 H'(y;a_2,\underline{\theta}_2)\exp\{-\lambda_2 H(y;a_2,\underline{\theta}_2)\};\ y>a_2\geq 0,\ \lambda_2>0.$$

We assume that λ_1 and λ_2 are unknown, but a_1, a_2, $\underline{\theta}_1$ and $\underline{\theta}_2$ are known. Let n items on X and m items on Y are put on a life test and the truncation numbers for X and Y are r_1 and r_2, respectively. Let us denote by

$$S_{r_1}=\sum_{i=1}^{r_1}G(x_{(i)};a_1,\underline{\theta}_1)+(n-r_1)G(x_{(r_1)};a_1,\underline{\theta}_1)$$

and

$$T_{r_2}=\sum_{j=1}^{r_2}H(y_{(j)};a_2,\underline{\theta}_2)+(m-r_2)H(y_{(r_2)};a_2,\underline{\theta}_2).$$

In what follows, we obtain the UMVUE of 'P'.

Theorem 2: The UMVUE of 'P' is given by

$$\hat{P}_{II}=\begin{cases}(r_2-1)\displaystyle\int_{1-\frac{H(G^{-1}(S_{r_1}))}{T_{r_2}}}^{1} z^{r_2-1}\left[1-\frac{G(H^{-1}((1-z)T_{r_2}))}{S_{r_1}}\right]^{r_1-1}dz, & G^{-1}(S_{r_1})<H^{-1}(T_{r_2})\\[2ex] (r_2-1)\displaystyle\int_0^1 z^{r_2-2}\left[1-\frac{G(H^{-1}((1-z)T_{r_2}))}{S_{r_1}}\right]^{r_1-1}dz, & H^{-1}(T_{r_2})<G^{-1}(S_{r_1}).\end{cases}$$

Proof: It follows from Lemma 3 that the UMVUES of $f_1(x;a_1,\lambda_1,\underline{\theta}_1)$ and $f_2(y;a_2,\lambda_2,\underline{\theta}_2)$ at specified points 'x' and 'y', respectively, are

$$\hat{f}_{1II}(x;a_1,\lambda_1,\underline{\theta}_1)=\begin{cases}\dfrac{(r_1-1)G'(x;a_1,\underline{\theta}_1)}{S_{r_1}}\left[1-\dfrac{G(x;a_1,\underline{\theta}_1)}{S_{r_1}}\right]^{(r_1-2)}, & G(x;a_1,\underline{\theta}_1)<S_{r_1}\\ 0, & \text{otherwise}\end{cases}\tag{3.6}$$

and

$$\hat{f}_{2II}(y;a_2,\lambda_2,\underline{\theta}_2)=\begin{cases}\dfrac{(r_2-1)H'(y;a_2,\underline{\theta}_2)}{T_{r_2}}\left[1-\dfrac{H(y;a_2,\underline{\theta}_2)}{T_{r_2}}\right]^{(r_2-2)}, & H(y;a_2,\underline{\theta}_2)<T_{r_2}\\ 0, & \text{otherwise.}\end{cases}\tag{3.7}$$

From the arguments similar to those adopted in proving Theorem1, it can be shown that the UMVUE of 'P' is given by

$$\hat{P}_{II} = \int_{y=a_2}^{\infty}\int_{x=y}^{\infty} \hat{f}_{1II}(x;a_1,\lambda_1,\underline{\theta}_1)\hat{f}_{2II}(y;a_2,\lambda_2,\underline{\theta}_2)\, dx\, dy,$$

which on using (3.6) and (3.7) gives that

$$\hat{P}_{II} = \frac{(r_1-1)(r_2-1)}{S_{r_1}T_{r_2}}\int_{y=a_2}^{H^{-1}(T_{r_2})}\int_{x=y}^{G^{-1}(S_{r_1})} G'(x;a_1,\underline{\theta}_1)H'(y;a_2,\underline{\theta}_2)\left[1-\frac{G(x;a_1,\underline{\theta}_1)}{S_{r_1}}\right]^{r_1-2}\left[1-\frac{H(y;a_2,\underline{\theta}_2)}{T_{r_2}}\right]^{r_2-2} dxdy$$

$$= \frac{(r_2-1)}{T_{r_2}}\int_{y=a_2}^{\min\{G^{-1}(S_{r_1}),H^{-1}(T_{r_2})\}}\left[1-\frac{G(y;a_1,\underline{\theta}_1)}{S_{r_1}}\right]^{r_1-1} H'(y;a_2,\underline{\theta}_2)\left[1-\frac{H(y;a_2,\underline{\theta}_2)}{T_{r_2}}\right]^{r_2-2} dy. \tag{3.8}$$

The theorem now follows from (3.8).

Corollary 1: In the case when $a_1 = a_2 = a$, say, $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$, say, $G(x;\ a,\ \underline{\theta}) = H(x;\ a, \underline{\theta})$, but $\lambda_1 \neq \lambda_2$,

$$\hat{P}_{II} = \begin{cases} (r_2-1)\sum_{i=0}^{r_2-2}(-1)^i\binom{r_2-2}{i}\left(\frac{S_{r_1}}{T_{r_2}}\right)^{i+1} B(i+1,\ r_1), & S_{r_1} < T_{r_2} \\ (r_2-1)\sum_{i=0}^{r_1-1}(-1)^i\binom{r_1-1}{i}\left(\frac{T_{r_2}}{S_{r_1}}\right)^{i} B(i+1,\ r_2-1), & T_{r_2} < S_{r_1}. \end{cases}$$

Proof: From Theorem 2, for $S_{r_1} < T_{r_2}$,

$$\hat{P}_{II} = (r_2-1)\int_{1-\frac{S_{r_1}}{T_{r_2}}}^{1} z^{r_2-2}\left[1-\frac{T_{r_2}}{S_{r_1}}(1-z)\right]^{r_1-1} dz$$

$$= (r_2-1)\int_{0}^{S_{r_1}/T_{r_2}} (1-u)^{r_2-2}\left[1-\frac{T_{r_2}}{S_{r_1}}u\right]^{r_1-1} du$$

$$= (r_2-1)\sum_{i=0}^{r_2-2}(-1)^i\binom{r_2-2}{i}\left(\frac{S_{r_1}}{T_{r_2}}\right)^{i+1}\int_0^1 v^i(1-v)^{r_1-1}dv$$

and the first assertion follows. Furthermore, for $T_{r_2} < S_{r_1}$,

$$\hat{P}_{II} = (r_2-1)\int_0^1 z^{r_2-2}\left[1-\frac{T_{r_2}}{S_{r_1}}(1-z)\right]^{r_1-1} dz$$

$$= (r_2-1)\sum_{i=0}^{r_1-1}(-1)^i\binom{r_1-1}{i}\left(\frac{T_{r_2}}{S_{r_1}}\right)^{i}\int_0^1 u^{r_2-2}(1-u)^i\, du$$

and the second assertion follows.

Now we consider estimation based on type I censored data. Let $a \le X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ be the failure times of n items under test from (2.1). The test begins at time $X_{(0)} = a$ and the system operates till $X_{(1)} = x_{(1)}$ when the first failure occurs. The failed item is replaced by a new one and the system operates till the second failure occurs at time $X_{(2)} = x_{(2)}$, and so on. The experiment is terminated at time t_o.

Lemma 4: If $N(t_o)$ be the number of failures during the interval [0, t_o], then

$$P[N(t_o)=r|t_o]=\frac{\{n\lambda G(t_o;a,\underline{\theta})\}^r}{r!}\exp\{-n\lambda G(t_o;a,\underline{\theta})\}.$$

Proof: Let us make the transformations $W_1 = G(X_{(1)};a,\underline{\theta})$, $W_2 = G(X_{(2)};a,\underline{\theta}) - G(X_{(1)};a,\underline{\theta})$, …, $W_n = G(X_{(n)};a,\underline{\theta}) - G(X_{(n-1)};a,\underline{\theta})$. The pdf of W_1 is

$$h(w_1) = n\lambda \exp(-n\lambda w_1).$$

Moreover, W_2, … , W_n are independent and identically distributed as W_1. Using the monotonicity property of G(x; a, $\underline{\theta}$),

$$\begin{aligned} P[N(t) = r|t_o] &= P[X_{(r)} \le t_o] - P[X_{(r+1)} \le t_o] \\ &= P[G(X_{(r)};a,\underline{\theta}) \le G(t_o;a,\underline{\theta})] - P[G(X_{(r+1)};a,\underline{\theta}) \le G(t_o;a,\underline{\theta})] \\ &= P[W_1 + W_2 + \ldots + W_r \le G(t_o;a,\underline{\theta})] - P[W_1 + W_2 + \ldots + W_{r+1} \le G(t_o;a,\underline{\theta})]. \end{aligned} \tag{3.9}$$

From the additive property of exponentially distributed rv's [see Johnson and Kotz (1970), p.170], $U = n\lambda \sum_{i=1}^{r} W_i$ follows gamma distribution with pdf

$$h(u)=\frac{1}{\Gamma(r)}u^{r-1}e^{-u}\ ;u>0 \tag{3.10}$$

Using (3.10) and a result of Patel, Kapadia and Owen (1976, p.244), we obtain from (3.9) that

$$\begin{aligned} P[N(t_o)=r|t_o] &= \frac{1}{\Gamma(r+1)}\int_{n\lambda G(t_o;a,\underline{\theta})}^{\infty} e^{-u}u^r\,du - \frac{1}{\Gamma(r)}\int_{n\lambda G(t_o;a,\underline{\theta})}^{\infty} e^{-u}u^{r-1}\,du \\ &= \exp\{-n\lambda G(t_o;a,\underline{\theta})\}\left[\sum_{j=0}^{r}\frac{\{n\lambda G(t_o;a,\underline{\theta})\}^j}{j!} - \sum_{j=0}^{r-1}\frac{\{n\lambda G(t_o;a,\underline{\theta})\}^j}{j!}\right] \end{aligned}$$

and the lemma follows.

In the following lemma, we derive the UMVUE of λ^q, where q is a positive integer.

Lemma 5: For q to be a positive integer, the UMVUE of λ^q is given by

$$\hat{\lambda}_I^q = \begin{cases} \dfrac{r!}{(r-q)!}[nG(t_o;a,\underline{\theta})]^{-q} (q \le r) \\ 0, \ \text{otherwise.} \end{cases}$$

Proof: It follows from Lemma 4 and Fisher-Neyman factorization theorem [see Rohatgi (1976, p.341)] that r is sufficient for estimating λ. Moreover, since the distribution of r belongs to exponential family, it is also complete [see Rohatgi (1976, p.347)]. The lemma now follows from the result that the q^{th} factorial moment of distribution of r is given by

$E\{r(r-1)\ldots(r-q+1)\}=\{n\lambda G(t_o;a,\underline{\theta})\}^q$.

In the following lemma, we obtain the UMVUE of the sampled pdf (2.1) at a specified point 'x'.

Lemma 6: The UMVUE of $f(x;a,\lambda,\underline{\theta})$ at a specified point 'x' is

$$\hat{f}_I(x;a,\lambda,\underline{\theta})=\begin{cases}\dfrac{rG'(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\left[1-\dfrac{G(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right]^{r-1} & ;\ G(x;a,\underline{\theta})<nG(t_o;a,\underline{\theta})\\ 0,\ \text{otherwise.}\end{cases}$$

Proof: Using Lemma 1 of Chaturvedi and Tomer (2002) and Lemma 5, from (3.4), the UMVUE of $f(x;a,\lambda,\underline{\theta})$ at a specified point 'x' is

$$\hat{f}_I(x;a,\lambda,\underline{\theta})=G'(x;a,\underline{\theta})\sum_{i=0}^{r-1}\frac{(-1)^i}{i!}G^i(x;a,\underline{\theta})\,\hat{\lambda}_I^{i+1}$$

$$=G'(x;a,\underline{\theta})\sum_{i=0}^{r-1}\left\{\frac{(-1)^i}{i!}G^i(x;a,\underline{\theta})\right\}\left\{\frac{r!}{(r-i-1)}\right\}\{nG(t_o;a,\underline{\theta})\}^{-(i+1)}$$

$$=\frac{rG'(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\left\{\frac{G(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right\}^i$$

and the lemma follows.

In the following theorem, we derive the UMVUE of R(t).

Theorem 3: The UMVUE of R(t) is given by

$$\hat{R}_I(t)=\begin{cases}\left[1-\dfrac{G(t;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right]^r; & G(t;a,\underline{\theta})<nG(t_o;a,\underline{\theta})\\ 0,\ \text{otherwise.}\end{cases}$$

Proof: From the arguments similar to those adopted in the proof of Theorem 1, using Lemma 6,

$$\hat{R}_I(t)=\int_t^{\infty}\hat{f}_I(x;a,\lambda,\underline{\theta})dx$$

$$=\frac{r}{nG(t_o;a,\underline{\theta})}\int_t^{G^{-1}(nG(t_o;a,\underline{\theta}))}G'(x;a,\underline{\theta})\left[1-\frac{G(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right]^{r-1}dx$$

$$=r\int_{\frac{G(t;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}}^{1}(1-y)^{r-1}dy$$

and the theorem follows.

In what follows, we obtain UMVUE of 'P'. Suppose n items on X and m items on Y are put through a life test and t_o and t_{oo} are their truncation times, respectively. Let r_1 items on X and r_2 items on Y fail before times t_o and t_{oo}, respectively.

Theorem 4: The UMVUE of 'P' is given by

$$\hat{P}_I = \begin{cases} r_2 \int_0^1 \left\{1 - \dfrac{G(H^{-1}(mH(t_{oo};a_2,\underline{\theta}_2)z))}{nG(t_o;a_1,\underline{\theta}_1)}\right\}^{r_1} (1-z)^{r_2-1} dz, \\ \quad H^{-1}(mH(t_{oo};a_2,\underline{\theta}_2)) < G^{-1}(nG(t_o;a_1,\underline{\theta}_1)) \\ r_2 \int_0^{H(G^{-1}(nG(t_o;a_1,\underline{\theta}_1)))\{mH(t_{oo};a_2,\underline{\theta}_2)\}^{-1}} \left\{1 - \dfrac{G(H^{-1}(mH(t_{oo};a_2,\underline{\theta}_2)z))}{nG(t_o;a_1,\underline{\theta}_1)}\right\}^{r_1} (1-z)^{r_2-1} dz, \\ \quad G^{-1}(nG(t_o;a_1,\underline{\theta}_1)) < H^{-1}(mH(t_{oo};a_2,\underline{\theta}_2)). \end{cases}$$

Proof: Using the arguments similar to those applied in the proofs of Theorem 1 and Lemma 6,

$$\hat{P}_I = \int_{y=a_2}^{\infty} \int_{x=y}^{\infty} \hat{f}_{1I}(x;a_1,\lambda_1,\underline{\theta}_1)\, \hat{f}_{2I}(y;a_2,\lambda_2,\underline{\theta}_2)\, dx\, dy$$

$$= \frac{r_1 r_2}{nmG(t_o;a_1,\underline{\theta}_1)H(t_{oo};a_2,\underline{\theta}_2)} \int_{y=a_2}^{\infty} \int_{x=y}^{G^{-1}(nG(t_o;a_1,\underline{\theta}_1))} G'(x;a_1,\underline{\theta}_1)\, H'(y;a_2,\underline{\theta}_2)$$

$$\cdot \left[1 - \frac{G(x;a_1,\underline{\theta}_1)}{nG(t_o;a_1,\underline{\theta}_1)}\right]^{r_1-1} \left[1 - \frac{H(y;a_2,\underline{\theta}_2)}{mH(t_{oo};a_2,\underline{\theta}_2)}\right]^{r_2-1} dx\, dy$$

$$= \frac{r_1 r_2}{mH(t_{oo};a_2,\underline{\theta}_2)} \int_{y=a_2}^{\infty} \int_{u=\frac{G(y;a_1,\underline{\theta}_1)}{nG(t_o;a_1,\underline{\theta}_1)}}^{1} (1-u)^{r_1-1} H'(y;a_2,\underline{\theta}_2) \left[1 - \frac{H(y;a_2,\underline{\theta}_2)}{mH(t_{oo};a_2,\underline{\theta}_2)}\right]^{r_2-1} du\, dy$$

$$= \frac{r_2}{mH(t_{oo};a_2,\underline{\theta}_2)} \int_{a_2}^{\min\{G^{-1}(nG(t_o;a_1,\underline{\theta}_1)),\ H^{-1}(mH(t_{oo};a_2,\underline{\theta}_2))\}} H'(y;a_2,\underline{\theta}_2)$$

$$\cdot \left[1 - \frac{G(y;a_1,\underline{\theta}_1)}{nG(t_o;a_1,\underline{\theta}_1)}\right]^{r_1} \left[1 - \frac{H(y;a_2,\underline{\theta}_2)}{mH(t_{oo};a_2,\underline{\theta}_2)}\right]^{r_2-1} dy. \tag{3.11}$$

The theorem now follows from (3.11).

Corollary 2: In the case when $a_1 = a_2 = a$, say, $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$, say, $G(x;a_1,\underline{\theta}_1) = H(x;a_2,\underline{\theta}_2)$, $t_o = t_{oo}$, but $\lambda_1 \neq \lambda_2$,

$$\hat{P}_I = \begin{cases} r_2 \sum_{i=0}^{r_1} (-1)^i \binom{r_1}{i} \left(\dfrac{m}{n}\right)^i B(i+1, r_2), & m<n \\ r_2 \sum_{i=0}^{r_2-1} (-1)^i \binom{r_2-1}{i} \left(\dfrac{n}{m}\right)^{i+1} B(i+1, r_1+1), & n<m. \end{cases}$$

Proof: From Theorem 4, for m<n,

$$\hat{P}_I = r_2 \int_0^1 \left(1 - \frac{m}{n} z\right)^{r_1} (1-z)^{r_2-1} dz$$

$$= r_2 \sum_{i=0}^{r_1} (-1)^i \binom{r_1}{i} \left(\frac{m}{n}\right)^i \int_0^1 z^i (1-z)^{r_2-1} dz$$

and the first assertion follows. For n<m,

$$\hat{P}_I = r_2 \int_0^{n/m} \left(1-\frac{m}{n}z\right)^{r_1} (1-z)^{r_2-1} dz$$

$$= r_2 \left(\frac{n}{m}\right) \int_0^1 (1-u)^{r_1} \left(1-\frac{n}{m}u\right)^{r_2-1} du$$

$$= r_2 \sum_{i=0}^{r_2-1} (-1)^i \binom{r_2-1}{i} \left(\frac{n}{m}\right)^{i+1} \int_0^1 u^i (1-u)^{r_1} du$$

and the second assertion follows.

4. MLE's of R(t) and 'P' under Type I and Type II Censorings

In order to compare the performance of UMVUES and MLES, we consider the case when 'a' and $\underline{\theta}$ are known, but λ is unknown.

We first consider estimation based on type II censoring.

Lemma 7: The MLE of λ is

$$\hat{\lambda}_{II} = \frac{r}{S_r}.$$

Proof: From (3.3), the log-likelihood is

$$\log L(\lambda \mid \underline{x}) = \log\{n(n-1)...(n-r+1)\} + r\log\lambda + \sum_{i=1}^{r} \log\{G'(x_{(i)};a,\underline{\theta})\} - \lambda S_r . \quad (4.1)$$

The result now follows on differentiating (4.1) with respect to λ, equating the differential coefficient to zero and solving the equation for λ.

Lemma 8: The MLE of $f(x; a, \lambda, \underline{\theta})$ at a specified point 'x' is

$$\hat{f}_{II}(x; a, \lambda, \underline{\theta}) = \left(\frac{r}{S_r}\right) G'(x;a,\underline{\theta}) \exp\left\{-\left(\frac{r}{S_r}\right) G(x;a,\underline{\theta})\right\}.$$

Proof: The proof follows from (2.1), Lemma 7 and one-to-one property of the MLE.

Theorem 5: The MLE of R(t) is given by

$$\hat{R}_{II}(t) = \exp\left\{-\left(\frac{r}{S_r}\right) G(t;a,\underline{\theta})\right\}.$$

Proof: From one- to-one property of MLE,

$$\hat{R}_{II}(t) = \int_t^\infty \hat{f}_{II}(x; a, \lambda, \underline{\theta})\, dx ,$$

which, on using Lemma 8, gives that

$$\hat{R}_{II}(t) = \left(\frac{r}{S_r}\right) \int_t^\infty G'(x;a,\underline{\theta}) \exp\left\{-\left(\frac{r}{S_r}\right) G(x;a,\underline{\theta})\right\} dx$$

$$= \int_{\left(\frac{r}{S_r}\right) G(t;a,\underline{\theta})}^{\infty} e^{-y}\, dy$$

and the theorem follows.

Theorem 6: The MLE of 'P' is given by

$$\hat{P}_{II} = \int_0^{\infty} \exp\left\{-\left(\frac{r_1}{S_{r_1}}\right) G\left(H^{-1}\left(\frac{T_{r_2}}{r_2}v\right)\right)\right\} e^{-v}\, dv .$$

Proof: From one-to-one property of MLE,

$$\hat{P}_{II} = \int_{y=a_2}^{\infty} \int_{x=y}^{\infty} \hat{f}_{1II}(x;a_1,\lambda_1,\underline{\theta}_1)\, \hat{f}_{2II}(y;a_2,\lambda_2,\underline{\theta}_2)\, dx\, dy ,$$

which, on using Lemma 8, gives that

$$\hat{P}_{II} = \left(\frac{r_1 r_2}{S_{r_1} T_{r_2}}\right) \int_{y=a_2}^{\infty} \int_{x=y}^{\infty} \left\{G'(x;a_1,\underline{\theta}_1) \exp\left(-\left(\frac{r_1}{S_{r_1}}\right) G(x;a_1,\underline{\theta}_1)\right)\right\}$$

$$.\left\{H'(y;a_2,\underline{\theta}_2) \exp\left(-\left(\frac{r_2}{T_{r_2}}\right) H(y;a_2,\underline{\theta}_2)\right)\right\} dx\, dy$$

$$= \left(\frac{r_2}{T_{r_2}}\right) \int_{y=a_2}^{\infty} \int_{u=\left(\frac{r_1}{S_{r_1}}\right) G(y;a_1,\underline{\theta}_1)}^{\infty} e^{-u}\, H'(y;a_2,\underline{\theta}_2) \exp\left\{-\left(\frac{r_2}{T_{r_2}}\right) H(y;a_2,\underline{\theta}_2)\right\} du\, dy$$

$$= \left(\frac{r_2}{T_{r_2}}\right) \int_{y=a_2}^{\infty} \exp\left\{-\left(\frac{r_1}{S_{r_1}}\right) G(y;a_1,\underline{\theta}_1)\right\} H'(y;a_2,\underline{\theta}_2) \exp\left\{-\left(\frac{r_2}{T_{r_2}}\right) H(y;a_2,\underline{\theta}_2)\right\} dy$$

and the theorem follows.

Corollary 3: In the case when $a_1 = a_2 = a$, say, $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$, say, $G(x;a_1,\underline{\theta}_1) = H(x;a_2,\underline{\theta}_2)$, but $\lambda_1 \neq \lambda_2$,

$$\hat{P}_{II} = \left(\frac{r_2 S_{r_1}}{r_2 S_{r_1} + r_1 T_{r_2}}\right).$$

Now we consider estimation base on type I censoring.

Lemma 9: The MLE of λ is

$$\hat{\lambda}_I = \frac{r}{nG(t_o;a,\underline{\theta})} .$$

Proof: From Lemma 4, the log-likelihood is

$$\log L(\lambda \mid r) = r\log n + r\log G(t_o;a,\underline{\theta}) - \log r! + r\log\lambda - n\lambda G(t_o;a,\underline{\theta})$$

and the result follows.

Lemma 10: The MLE of $f(x;a,\lambda,\underline{\theta})$ at a specified point 'x' is

$$\hat{f}_I(x;a,\lambda,\underline{\theta}) = \left\{\frac{rG'(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right\} \exp\left\{-\frac{rG(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right\}.$$

Proof: The Lemma follows from (2.1) and Lemma 9.

Theorem 7: The MLE of R(t) is given by

$$\hat{R}_I(t) = \exp\left\{-\frac{rG(t;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right\}.$$

Proof: From Lemma 10,

$$\hat{R}_I(t) = \int_t^{\infty} \hat{f}_I(x;a,\lambda,\underline{\theta})\,dx$$

$$= \left\{\frac{r}{nG(t_o;a,\underline{\theta})}\right\}\int_t^{\infty} G'(x;a,\underline{\theta})\exp\left\{-\frac{rG(x;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}\right\} dx$$

$$= \int_{\frac{rG(t;a,\underline{\theta})}{nG(t_o;a,\underline{\theta})}}^{\infty} e^{-y}\,dy$$

and the theorem follows.

Theorem 8: The MLE of 'P' is given by

$$\hat{P}_I = \int_0^{\infty} \exp\left\{-\frac{r_1 G\left(H^{-1}\left(\frac{m}{r_2}H(t_{oo};a_2,\underline{\theta}_2)v\right)\right)}{nG(t_o;a_1,\underline{\theta}_1)}\right\} e^{-v}dv.$$

Proof: From Lemma 10,

$$\hat{P}_I = \int_{y=a_2}^{\infty}\int_{x=y}^{\infty} \hat{f}_{1I}(x;a_1,\lambda_1,\underline{\theta}_1)\,\hat{f}_{2I}(y;a_2,\lambda_2,\underline{\theta}_2)\,dx\,dy$$

$$= \left\{\frac{r_1 r_2}{nmG(t_o;a_1,\underline{\theta}_1)H(t_{oo};a_2,\underline{\theta}_2)}\right\}\int_{y=a_2}^{\infty}\int_{x=y}^{\infty} G'(x;a_1,\underline{\theta}_1)$$

$$.\exp\left\{-\frac{r_1 G(x;a_1,\underline{\theta}_1)}{nG(t_o;a_1,\underline{\theta}_1)}\right\} H'(y;a_2,\underline{\theta}_2)\exp\left\{-\frac{r_2 H(y;a_2,\underline{\theta}_2)}{mH(t_{oo};a_2,\underline{\theta}_2)}\right\} dxdy$$

$$= \left\{\frac{r_2}{mH(t_{oo};a_2,\underline{\theta}_2)}\right\}\int_{y=a_2}^{\infty}\int_{u=\frac{r_1 G(y;a_1,\underline{\theta}_1)}{nG(t_o;a_1,\underline{\theta}_1)}}^{\infty} e^{-u}\, H'(y;a_2,\underline{\theta}_2)\exp\left\{-\frac{r_2 H(y;a_2,\underline{\theta}_2)}{mH(t_{oo};a_2,\underline{\theta}_2)}\right\} dudy$$

$$= \left\{\frac{r_2}{mH(t_{oo};a_2,\underline{\theta}_2)}\right\}\int_{a_2}^{\infty} \exp\left\{-\frac{r_1 G(y;a_1,\underline{\theta}_1)}{nG(t_o;a_1,\underline{\theta}_1)}\right\} H'(y;a_2,\underline{\theta}_2)\exp\left\{-\frac{r_2 H(y;a_2,\underline{\theta}_2)}{mH(t_{oo};a_2,\underline{\theta}_2)}\right\} dy$$

and the theorem follows.

Corollary 4: In the case when $a_1 = a_2 = a$, say, $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$, say, $G(x;a_1,\underline{\theta}_1) = H(x;a_2,\underline{\theta}_2)$, $t_o = t_{oo}$, but $\lambda_1 \neq \lambda_2$,

$$\hat{P}_I = \frac{r_2 n}{r_2 n + r_1 m}.$$

Remarks 1:

(i) In the literature, researchers have dealt with the estimation of R(t) and 'P', separately. If we look at the proofs of Theorems 1-8, we observe that the UMVUE(S) / MLE(S) of power(s) of parameter(s) is (are) used to obtain UMVUE(S) / MLE(S) of the sampled pdf(s), which is (are) subsequently used to estimate R(t) and 'P'. Thus, for both the estimation problems, the basic role is played by the estimator(s) of power(s) of parameter(s). In this way, we have justified estimation of power(s) of parameter(s).

(ii) We have established an interrelationship between the estimation of R(t) and 'P'.

(iii) In the literature, the researchers have derived the UMVUES / MLES of 'P' for the case when X and Y follow the same distribution (may be with different parameters). We have obtained these estimators for all the three situations, when X and Y follow the same distribution having all the parameters same other than λ's, when X and Y have the same distribution with different parameters and when X and Y follow different distributions.

(iv) In the present approaches of obtaining UMVUES and MLES, one does not need the expressions of R(t) and 'P'.

(v) The problems of obtaining MLES when more parameters are unknown can be dealt on similar lines. One just needs as many differentials (with respect to unknown parameters) of likelihood function as the number of unknown parameters and their simultaneous solutions. The MLE of any parametric function can be obtained by plugging the MLES in place of unknown parameters.

(vi) It follows from Lemma 2 that $\mathrm{Var}(\hat{\lambda}_{II}) = \dfrac{\lambda^2}{(r-2)} \to 0$ as $r \to \infty$. Moreover, from Lemma7, $E(\tilde{\lambda}_{II}) = \dfrac{r\lambda}{(r-1)} \to \lambda$ as $r \to \infty$ and $\mathrm{Var}(\tilde{\lambda}_{II}) = \dfrac{r^2\lambda^2}{(r-1)^2(r-2)} \to 0$ as $r \to \infty$. Thus, $\hat{\lambda}_{II}$ and $\tilde{\lambda}_{II}$ are consistent estimators of λ. Since $\hat{f}_{II}(x; a, \lambda, \underline{\theta})$, $\tilde{f}_{II}(x; a, \lambda, \underline{\theta})$, $\hat{R}_{II}(t)$, $\tilde{R}_{II}(t)$, $\hat{P}_{II}$ and $\tilde{P}_{II}$ are continuous functions of consistent estimators, they are also consistent estimators.

5. Simulation Studies

We have shown under Remarks 1(vi) that $\hat{\lambda}_{II}$, $\tilde{\lambda}_{II}$, $\hat{f}_{II}(x; a, \lambda, \underline{\theta})$, $\tilde{f}_{II}(x; a, \lambda, \underline{\theta})$, $\hat{R}_{II}(t)$, $\tilde{R}_{II}(t)$, $\hat{P}_{II}$ and $\tilde{P}_{II}$ are consistent estimators. In order to verify these results, we have drawn sample of size n= 50 from (2.1) with $G(x;p) = x^p$, a=0, p=2 and λ =1. In

Fig.1 and Fig.2, respectively, we have plotted $\hat{f}(x;a,\lambda,\underline{\theta})$ and $\tilde{f}(x;a,\lambda,\underline{\theta})$ for different values of r=5(5)30 and 50 under type II censoring. We conclude from the figures that as r increases, the curves of $\hat{f}_{II}(x;a,\lambda,\underline{\theta})$ and $\tilde{f}_{II}(x;a,\lambda,\underline{\theta})$ come close to the curve of $f(x;a,\lambda,\underline{\theta})$. This justifies the consistency property of the estimators.

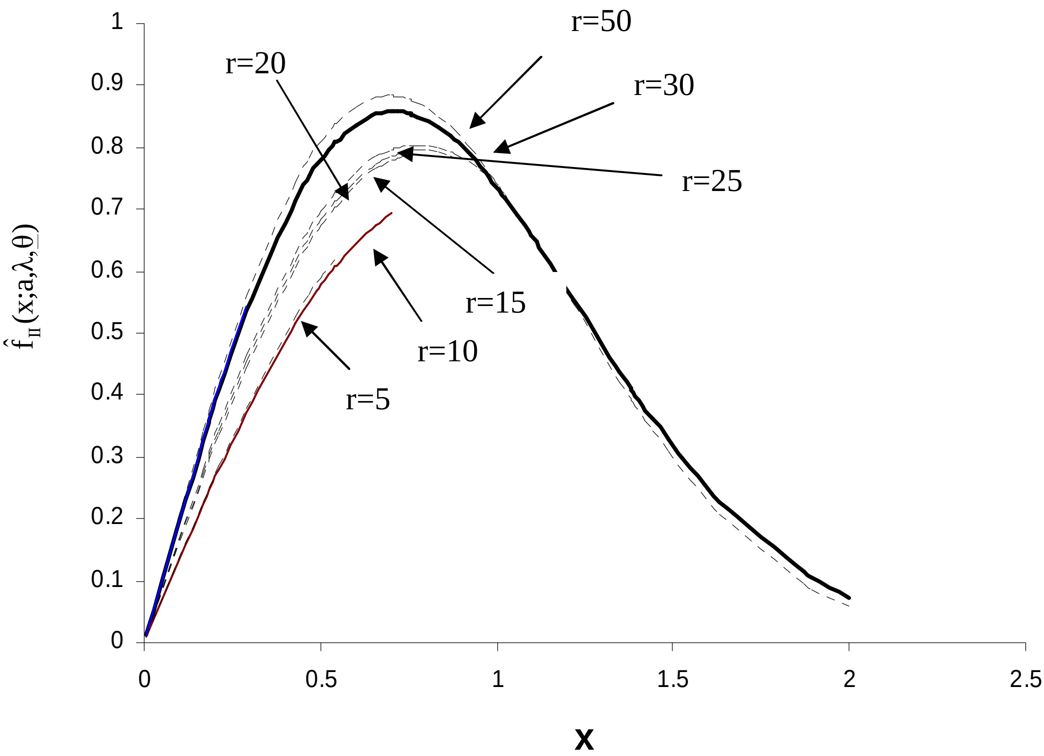

Fig.1: The curves of $f(x;a,\lambda,\underline{\theta})$. (bold) and $\hat{f}_{II}(x;a,\lambda,\underline{\theta})$ (dotted)

For the case when λ is unknown, we have conducted a simulation experiments using bootstrap resampling technique of the following sample of size 50, generated from (2.1) with $G(x;a,\underline{\theta}) = x$, a=0 and λ =0.0004.

49.2611	251.1228	270.5831	313.1192	360.7416	383.3871	411.4322
534.8131	566.5359	683.0055	772.1821	793.6235	795.7669	872.7322
936.7485	971.3557	1059.5648	1238.8516	1271.1188	1342.7648	1404.8366
1410.5900	1413.3407	1525.9684	1568.4431	626.5130	1662.05	1851.8340
1863.1614	1875.8562	1962.0475	2169.4619	2230.3038	2442.2024	2758.2105
2784.7300	2795.5681	2932.7959	3913.2557	4080.1081	4209.9014	4653.3542
4748.0918	5132.9441	5401.5366	5408.7457	5444.3025	7241.4239	11681.5470
11705.1838.						

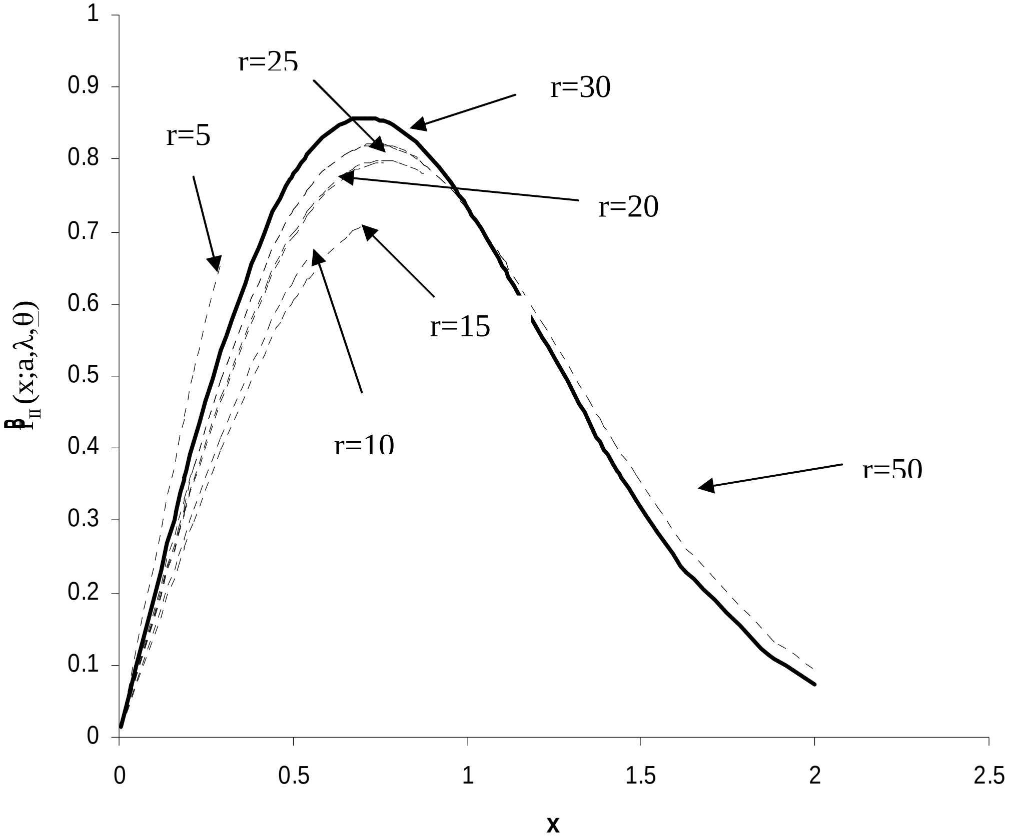

Fig.2: The curves of $f(x;a,\lambda,\underline{\theta})$. **(bold) and** $\hat{f}_{II}(x;a,\lambda,\underline{\theta})$ **(dotted)**

Assuming that the data represents life spans of items in hours. For different values of t and r, we have computed $\hat{R}(t)$, $\hat{R}(t)$, bias, variance/ mean sum of squares (MSES), 95% confidence length and corresponding coverage percentage under type I and type II censorings. All the computations are based on 500 bootstrap replications and the results under type I and type II [for t_o = Maximum(x_r)] censorings are reported in Table 1 and Table 2 respectively.

r		**5**		**8**		**10**		**15**		**20**	
t	R(t)	$\tilde{R}_I$	$\hat{R}_I$	$\tilde{R}_I$	$\hat{R}_I$	$\tilde{R}_I$	$\hat{R}_I$	$\tilde{R}_I$	$\hat{R}_I$	$\tilde{R}_I$	$\hat{R}_I$
100	0.9604	0.9709 0.0104 0.00012 0.0089 72.7895	0.9708 0.0104 0.00012 0.009 72.7874	0.9671 0.0067 7e-05 0.0114 71.3972	0.967 0.0066 7e-05 0.0115 71.3954	0.9683 0.0079 8e-05 0.0078 60.8975	0.9683 0.0079 8e-05 0.0079 60.8832	0.9673 0.0069 5e-05 0.0055 71.196	0.9673 0.0069 5e-05 0.0055 71.1952	0.9698 0.0094 9e-05 0.0024 50.3915	0.9698 0.0094 9e-05 0.0024 50.386
400	0.8508	0.8888 0.038 0.00155 0.0325 72.2166	0.8875 0.0367 0.00146 0.0333 72.2082	0.8747 0.0239 0.00088 0.041 71.1796	0.8737 0.0229 0.00084 0.0417 71.1792	0.8792 0.0284 0.00105 0.0284 62.3227	0.8784 0.0276 0.00101 0.0288 62.2741	0.8751 0.0243 0.00064 0.0197 73.3461	0.8746 0.0238 0.00062 0.0199 73.3434	0.8842 0.0334 0.00116 0.0088 47.0879	0.8839 0.0331 0.00114 0.0089 47.065
500	0.8171	0.8625 0.0454 0.00222 0.0393 72.9396	0.8605 0.0434 0.00206 0.0405 72.9283	0.8472 0.0301 0.00133 0.0494 70.8813	0.8457 0.0286 0.00126 0.0504 70.8821	0.8512 0.0341 0.00151 0.0343 63.1625	0.8501 0.033 0.00144 0.0349 63.1079	0.8464 0.0293 0.00093 0.0238 73.0971	0.8456 0.0285 0.00089 0.0241 73.0955	0.8571 0.04 0.00168 0.0118 49.916	0.8566 0.0395 0.00164 0.0119 49.8934
600	0.7848	0.8375 0.0527 0.00299 0.0457 73.1803	0.8348 0.05 0.00273 0.0473 73.1667	0.819 0.0342 0.00174 0.0572 71.2108	0.8168 0.0321 0.00163 0.0586 71.212	0.8265 0.0417 0.00218 0.0399 62.7205	0.8249 0.0402 0.00207 0.0406 62.6612	0.8196 0.0349 0.0013 0.0276 72.2089	0.8185 0.0338 0.0012 0.028 72.2049	0.8311 0.0463 0.00224 0.0124 47.2747	0.8304 0.0456 0.00218 0.0126 47.2447
1000	0.6677	0.7436 0.076 0.00695 0.0669 59.8607	0.7368 0.0691 0.00613 0.0707 59.3552	0.7155 0.0479 0.00352 0.0826 71.6969	0.7103 0.0427 0.00315 0.0858 71.701	0.7262 0.0585 0.00427 0.0582 65.9724	0.7224 0.0547 0.00389 0.0598 65.9095	0.7185 0.0508 0.00275 0.04 71.2692	0.7158 0.0482 0.0025 0.0408 71.2643	0.7365 0.0688 0.00489 0.0183 52.4815	0.7348 0.0671 0.00466 0.0186 52.4427
1300	0.5915	0.6804 0.0889 0.00863 0.0789 72.5518	0.6698 0.0783 0.00696 0.0845 72.5343	0.6516 0.0601 0.00515 0.0965 71.1847	0.6516 0.0601 0.00515 0.0965 71.1847	0.6624 0.0709 0.0063 0.0684 63.0157	0.6566 0.0651 0.00561 0.0708 62.9366	0.6495 0.0581 0.00365 0.0469 71.3247	0.6454 0.054 0.0032 0.048 71.32	0.671 0.0796 0.00656 0.0217 51.8713	0.6683 0.0769 0.00615 0.0221 51.831
1500	0.5456	0.6421 0.0966 0.01013 0.0854 72.5543	0.6288 0.0832 0.00785 0.0922 72.533	0.6078 0.0622 0.00579 0.1037 70.9945	0.5979 0.0523 0.00486 0.109 70.996	0.623 0.0774 0.00719 0.0739 66.7749	0.6158 0.0702 0.00622 0.0767 66.7266	0.6074 0.0619 0.00413 0.0505 72.875	0.6023 0.0567 0.00353 0.0518 72.8696	0.6308 0.0853 0.00754 0.0235 52.6228	0.6274 0.0819 0.00698 0.024 52.5837
2000	0.4458	0.5557 0.1099 0.01305 0.0969 72.0098	0.5352 0.0894 0.00916 0.1062 71.9947	0.5184 0.0726 0.00737 0.1157 72.0225	0.5037 0.0579 0.00571 0.1228 72.0314	0.5315 0.0857 0.00878 0.0834 68.5306	0.5204 0.0746 0.00714 0.0873 68.5065	0.516 0.0702 0.00525 0.0565 72.3871	0.5082 0.0625 0.00425 0.0583 72.3839	0.5407 0.0949 0.00939 0.0269 50.3065	0.5355 0.0897 0.00845 0.0275 50.2683

Table 1: Simulation results for estimation of R(t) under Type I censoring

First row indicates the average estimates, the second row indicates the bias, the third row indicates MSE / Variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

r		**5**		**8**		**10**		**15**		**20**	
t	R(t)	$\tilde{R}_{II}$	$\hat{R}_{II}$	$\tilde{R}_{II}$	$\hat{R}_{II}$	$\tilde{R}_{II}$	$\hat{R}_{II}$	$\tilde{R}_{II}$	$\hat{R}_{II}$	$\tilde{R}_{II}$	$\hat{R}_{II}$
100	0.9604	0.9658	0.9691	0.9629	0.9646	0.9589	0.9605	0.9562	0.9576	0.9591	0.9601
		0.0054	0.0087	0.0024	0.0042	-.0015	1e-04	-0.0042	-.0028	-.0013	-3e-04
		5e-05	9e-05	1e-05	2e-05	1e-05	0	2e-05	1e-05	2e-05	2e-05
		0.0096	0.0087	0.0059	0.0056	0.0067	0.0064	0.0065	0.0063	0.0125	0.0122
		67.7959	67.7643	79.536	79.525	89.0394	89.0373	88.2244	88.2228	86.4103	86.406
400	0.8508	0.8707	0.882	0.859	0.8651	0.8452	0.8504	0.836	0.8406	0.846	0.8492
		0.0199	0.0312	0.0082	0.0143	-.0056	-4e-04	-0.0148	-0.0102	-0.0048	-0.0016
		0.00061	0.00116	0.00015	0.00028	9e-05	6e-05	0.00028	0.00017	0.00022	2e-04
		0.0331	0.0306	0.0222	0.0214	0.0244	0.0238	0.0245	0.0239	0.0467	0.0459
		71.4846	71.3959	75.7701	75.7177	87.8636	87.8528	87.1718	87.16	87.4368	87.425
500	0.8171	0.8419	0.8554	0.8272	0.8344	0.8105	0.8166	0.7998	0.8051	0.8126	0.8163
		0.0248	0.0383	0.0101	0.0173	-.0066	-5e-04	-0.0173	-0.012	-.0046	-8e-04
		0.00093	0.00174	0.00021	4e-04	0.00012	8e-05	0.00037	0.00021	3e-04	0.00028
		0.0396	0.0369	0.0276	0.0267	0.0295	0.0288	0.0266	0.0261	0.0572	0.0563
		71.1054	70.9898	80.8538	80.8105	89.0464	89.0375	88.5789	88.5727	88.9917	88.9786
600	0.7848	0.8119	0.8273	0.7967	0.8048	0.7775	0.7844	0.7651	0.7711	0.7791	0.7833
		0.0271	0.0425	0.0119	0.02	-0.0072	-3e-04	-0.0197	-0.0137	-.0057	-0.0014
		0.00125	0.00227	0.00026	0.00052	0.00014	8e-05	0.00049	0.00029	0.00036	0.00032
		0.0473	0.0444	0.0299	0.029	0.0297	0.0291	0.0316	0.031	0.0594	0.0586
		68.4233	68.2488	81.462	81.4136	88.5451	88.5352	87.8483	87.8423	86.7835	86.7658
1000	0.6611	0.7047	0.7256	0.6844	0.695	0.6577	0.6666	0.6392	0.6467	0.6607	0.6661
		0.0371	0.0579	0.0167	0.0274	-0.0099	-0.0011	-0.0285	-0.021	-0.007	-0.0016
		0.00245	0.00436	0.00053	0.00099	0.00026	0.00016	0.00102	0.00065	0.00061	0.00056
		0.0667	0.0642	0.0423	0.0417	0.0425	0.0421	0.0465	0.0461	0.0827	0.0822
		68.2689	67.9516	80.8597	80.7876	89.056	89.0427	88.9408	88.9332	89.7527	89.7345
1300	0.5915	0.6387	0.6617	0.6122	0.6237	0.5801	0.5895	0.5607	0.5685	0.5842	0.59
		0.0472	0.0702	0.0207	0.0323	-0.0113	-0.002	-0.0308	-0.023	-0.0073	-0.0015
		0.00334	0.006	0.00081	0.00142	0.00035	0.00023	0.00117	0.00075	0.00086	0.00081
		0.0779	0.0762	0.0468	0.0465	0.0487	0.0486	0.0474	0.0473	0.1	0.0999
		73.1837	72.9807	76.8806	76.7637	89.0756	89.0611	88.7632	88.7516	89.4526	89.4265
1500	0.5456	0.5944	0.6181	0.5681	0.5799	0.532	0.5414	0.5122	0.5199	0.5375	0.5433
		0.0489	0.0726	0.0225	0.0343	-0.0136	-0.0042	-0.0334	-0.0256	-0.0081	-0.0023
		0.00377	0.00664	0.00084	0.00151	0.00047	3e-04	0.00141	0.00095	0.00094	0.00089
		0.0815	0.0807	0.0504	0.0503	0.0562	0.0564	0.0532	0.0534	0.099	0.0992
		71.1787	70.9292	82.4645	82.3787	89.434	89.4178	87.7553	87.7359	88.6226	88.5911
2000	0.4458	0.5044	0.5281	0.4696	0.4809	0.4327	0.4413	0.411	0.4179	0.4361	0.4414
		0.0586	0.0823	0.0238	0.0351	-0.0131	-0.0045	-0.0347	-0.0279	-0.0097	-0.0044
		0.00491	0.00831	0.00104	0.00172	0.00049	0.00034	0.00153	0.00111	0.00126	0.00121
		0.0928	0.0944	0.0583	0.059	0.0588	0.0596	0.0593	0.0601	0.1135	0.1145
		73.7833	73.5675	81.1842	81.0605	88.3596	88.3405	89.7384	89.7273	88.5678	88.5335

Table 2: Simulation results for estimation of R(t) under type II censoring

* First row indicates the average estimates, the second row indicates the bias, the third row indicates MSE / variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

In order to estimate P, when X and Y follow the same distribution, for the case when λ_1 and λ_2 are unknown but the other parameters are known, we have conducted simulation experiments using bootstrap resampling technique for sample sizes (n, m) = (40, 30), (40, 40), (50, 40) and (50, 50) across different (r_1, r_2) = (10, 10), (10, 15), (15, 15) and (25, 25). The samples are generated from (2.1) with $G(x;p_1) = x^{p_1}$, $H(y;p_2) = y^{p_2}$, $a_1 = a_2 = 0$, $p_1 = p_2 = 2$, $\lambda_1 = 1$ and $1/\lambda_2 = 2$. The computations are based on 500 bootstrap replications. We have computed $\hat{P}_{II}$, ฿$_{II}$, bias/ variance, MSES, 95% confidence length and corresponding coverage percentage under type II censoring and the results are presented in Table 3.

r_1 r_2 / n, m	10, 10		10, 15		15, 15		25, 25	
P=0.33	฿$_{II}$	$\hat{P}_{II}$	฿$_{II}$	$\hat{P}_{II}$	฿$_{II}$	$\hat{P}_{II}$	฿$_{II}$	$\hat{P}_{II}$
50, 50	0.3435 0.0101 0.00043 0.0546 86.2663	0.3353 0.002 0.00036 0.0568 86.3479	0.3974 0.0641 0.00434 0.045 85.8556	0.3969 0.0636 0.0043 0.047 85.92	0.3566 0.0232 0.00073 0.0409 86.3665	0.3518 0.0184 0.00054 0.0421 86.3825	0.3404 0.007 2e-04 0.0399 90.0565	0.3373 0.004 0.00017 0.0405 90.0595
50, 40	0.3838 0.0504 0.00312 0.0907 87.4411	0.3774 0.0441 0.00258 0.0953 87.3628	0.4467 0.1134 0.01315 0.056 89.8405	0.4485 0.1151 0.01357 0.0587 89.8427	0.4035 0.0701 0.00517 0.0504 88.1438	0.4001 0.0667 0.00472 0.0521 88.1469	0.379 0.0456 0.00229 0.0479 89.9056	0.3766 0.0432 0.00209 0.0487 89.9088
40, 40	0.3065 -0.0268 0.00111 0.0731 82.4706	0.2972 -0.0362 0.00173 0.0755 82.3555	0.3644 0.0311 0.00114 0.0427 88.8581	0.3625 0.0292 0.00104 0.0444 88.8519	0.3382 0.0048 0.00043 0.0741 91.367	0.3329 -4e-04 0.00042 0.0757 91.406	0.3283 -0.005 0.00023 0.047 89.5136	0.3252 -0.0082 0.00028 0.0476 89.5144
40, 30	0.3347 0.0013 0.00013 0.0332 85.0491	0.3262 -0.0071 0.00019 0.0344 85.0051	0.4093 0.076 0.00596 0.0456 91.0323	0.4094 0.076 0.00597 0.0477 91.0297	0.3817 0.0484 0.00287 0.0802 89.1871	0.3776 0.0443 0.00252 0.0825 89.2229	0.3627 0.0293 0.00123 0.0599 88.1111	0.36 0.0267 0.00109 0.0609 88.1093

Table 3: Simulation results for estimation of P on Type-II censoring

***** First row indicates the average estimates, the second row indicates the bias, the third row indicates MSE / Variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

For the case when X and Y follow the different distributions, when all the parameters are unknown, the following samples (each of size 50) are generated from the distributions of X and Y. The samples are generated from (2.1) with $G(x;a,\underline{\theta}) = x^p$, p=3, $1/\lambda_1$ =2000, $H(y;a,\underline{\theta}) = \log(\frac{y}{a})$, a = 1000 and λ_2 =2.

X: 675.631, 696.691, 772.670, 814.196, 816.689, 919.121, 934.893, 955.100, 1008.590, 1103.114, 1140.375, 1336.660 ,1373.238, 1463.133, 1479.120, 1488.223, 1498.410, 1498.800, 1520.379, 1579.157, 1604.472, 1606.427, 1664.502, 1686.000, 1689.984, 1724.849, 1732.414, 1733.658 ,1821.057 ,1834.874, 1838.607, 1857.158, 1903.130, 2114.485 ,2164.578, 2169.304, 2176.311, 2230.114, 2261.894, 2327.029, 2355.698, 2360.320, 2408.375, 2542.200, 2573.111, 2784.734 ,2888.644 ,3014.000, 3087.838, 3203.454.

Y: 1002.180, 1007.831, 1014.529, 1032.043, 1035.586, 1049.101, 1050.847, 1060.742, 1077.574, 1132.504, 1141.820, 1143.473, 1162.935, 1165.665, 1191.798, 1222.211, 1227.363, 1251.771, 1282.447, 1368.168, 1413.533, 1414.386, 1506.587, 1509.096, 1525.781, 1558.127, 1589.134, 1676.492, 1709.565, 1725.422, 1825.827, 1839.870, 1863.492, 1940.611, 2012.242, 2069.430, 2276.403, 2439.215, 2591.761, 2690.088, 2837.542, 2841.985, 3103.432, 3617.702, 3855.677, 4030.920, 4058.424, 4565.295, 5286.151, 5693.574.

Let t_o =1500 be the truncation time for X and t_{oo} =1300 for Y, so we have r_1 =18 and r_2 =19. The MLES of the parameters of X are $\hat{p}$ = 0.72 and $\hat{\lambda}_1$ =0.002 and the MLES of the parameters of Y are $\hat{\lambda}_2$ = 1.4605 and $\hat{a}$ = 1002.18. We have P = 0.5142. Using Theorem 8, $\hat{P}_I$ = 0.6036 (with absolute error < 4.8e-05).

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