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ESTIMATION OF PARITY SPECIFIC FERTILITY RATES UNDER DIFFERENT FECUNDITY LEVELS

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Abstract

A number of attempts have been made to describe probabilistic behaviour of couple fertility in a given period of time (0,T). Singh et al. (1974), Singh (1964), Bhattacharya (1986), Pathak (1999) and Khan and Raeside (1998) have built the models to enhance the impact of birth intervals and fertility rates. Biswas (1980) considered the waiting time distribution as the convolution of several Poisson distributions with variable parameters λ_i. In the proposed study, it has been derived to estimate parity wise fertility rates under different fecundity levels based on renewal theory approach by considering the hazard rates of particular parity.

Key words: Fertility Analysis, Parity Progression Ratio, Birth Interval Distribution.

1. Introduction

Biswas (1980) considered the waiting time distribution for n (>1) birth as the convolution of several Poisson distributions with variable parameters λ_i (i = 1,2,...,n) where λ_i stands for the hazard rate of the i^{th} parity. Subsequently, the density function of the waiting time(t) would be given as below :

$$f_n(s_n \mid \lambda_1,\lambda_2,\lambda_3,\ldots\ldots,\lambda_n) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{\lambda_j}{\lambda_i - \lambda_j} \lambda_i e^{-\lambda_i s_n} \qquad (1)$$

$\lambda_1=\lambda$, $\lambda_2=\lambda e^{-\delta}$, $\lambda_3=\lambda e^{-2\delta}$, ..., $\lambda_n = \lambda e^{-(n-1)\delta}$

$$s_n = \sum_{i=1}^{n} t_i$$

where δ is considered to be very infinitesimally diminishing known constants.

The convolution of waiting time (t) between two consecutive births to a mother $t_1, t_2, t_3, \ldots\ldots, t_n$ is given by

$$s_n = t_1 + t_2 + t_3 + \ldots\ldots + t_n$$

While considering the parity specific fertility (hazard) rates in the foregoing model the individual differences are not taken into account. However, Brass (1958), Singh (1964) have assumed the variation of fecundability parameter λ given by

$$\Phi(\lambda) = \frac{a^k}{\Gamma(k)} e^{-a\lambda} \lambda^{k-1} : \quad 0 \le \lambda < \infty, \quad 0 < a, k < \infty \qquad (2)$$

It is presumed here that hazard rate λ follows a Gamma distribution with parameters k and a. This gives the waiting time distribution of n^{th} birth as weighted Poisson process with gradually diminishing intensity, given by

$$f_n(s_n) = \frac{a^k}{\Gamma(k)} \int_0^\infty \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-(j-1)\delta - (i-1)\delta}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} \times e^{-\lambda\left[e^{-(j-1)s_n + a}\right]} \lambda^k d\lambda$$

$$= \frac{ka^k}{\Gamma(k)} \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-\delta(i+j-2)}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} \times \frac{1}{[a + e^{-(j-1)\delta} s_n]^{k+1}} \quad (3)$$

Subsequently, the raw moments of the above distribution are deduced as

$$\mu'_r = E(S_n^r) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-\delta(i+j-2)}}{[e^{-(j-1)^\delta} - e^{-(i-1)^\delta}]} \times ka^k \int_0^\infty \frac{s_n^r ds_n}{[a + e^{-(j-1)\delta} s_n]^{k+1}}$$

$$= \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-\delta(i+j-2)}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} e^{-\delta(i-1)(r+1)} a^{r-k} \beta(r+1, k-r)$$

(4)

The successive recursive relationships between hazard rates are considered as:

$\lambda_1 = \lambda$

$\lambda_2 = \lambda_1 e^{-\delta_1} = \lambda e^{-\delta_1}$

$\lambda_i = \lambda_{i-1} e^{-\delta_{i-1}} = \lambda e^{-\delta_1} e^{-\delta_2} \ldots e^{-\delta_{i-1}} = \lambda e^{-(\delta_1+\delta_2+\ldots+\delta_{i-1})}$

And for $\delta_1 = \delta_2 = \ldots = \delta_i$, the above hazard rates reduce to the hazard rates of Biswas (1980). Further, λ varies from individual to individual conforming a two parameters family of Gamma Distribution. This gives the waiting time density function for the n[th] birth as

$$f_n(s_n) = \frac{a^k}{\Gamma(k)} \int_0^\infty \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-(\delta_1+\ldots+\delta_{j-1})} e^{-(\delta_1+\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots+\delta_{j-1})} - e^{-(\delta_1+\ldots+\delta_{i-1})}} \times e^{-\lambda\,[e^{-(\delta_1+\ldots+\delta_{i-1})} s_n + a]} \lambda^k d\lambda$$

$$= \frac{ka^k}{\Gamma(k)} \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-(\delta_1+\ldots+\delta_{j-1})} e^{-(\delta_1+\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots+\delta_{j-1})} - e^{-(\delta_1+\ldots+\delta_{i-1})}} \times \frac{1}{[a + e^{-(\delta_1+\ldots+\delta_{i-1})} s_n]^{k+1}} \quad (5)$$

and moments of the distribution will be

$$\mu'_r = E(S_n^r) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{e^{-(\delta_1+\ldots\ldots+\delta_{j-1})} e^{-(\delta_1+\ldots\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots\ldots+\delta_{j-1})} - e^{-(\delta_1+\ldots\ldots+\delta_{i-1})}} \times e^{-(\delta_1+\delta_2\ldots\ldots+\delta_{i-1})(r+1)} a^{r+k\beta(r+1,\, k-r)}$$

(6)

For n = 3, when i = 1,

$$\mu'_r = \prod_{j=2,3} \frac{e^{-\delta_1} e^{-(\delta_1+\delta_2)}}{e^{-\delta_1} - e^{-(\delta_1+\delta_2)}} a^{r-k\beta(r+1,\, k-r)} \quad (7)$$

for n = 3, when i = 2,

$$\mu'_r = \prod_{j=1,3} \frac{e^{-(\delta_1+\delta_2)} e^{-\delta_1}}{e^{-(\delta_1+\delta_2)} - e^{-\delta_1}} e^{-\delta_1(r+1)} a^{r+k\beta(r+1,\, k-r)}$$

$$= -\frac{e^{-\delta_1}\left(e^{-(\delta_1+\delta_2)}\right)}{e^{-\delta_1}\left(e^{-\delta_2} - 1\right)} e^{-\delta_1(r+1)} a^{r+k\beta(r+1,\, k-r)} \quad (8)$$

And for n = 3, when i = 2,

$$\mu'_r = \prod_{j=1,2} \frac{e^{-\delta_1}\left(e^{-(\delta_1+\delta_2)}\right)}{e^{-\delta 1} - e^{-(\delta_1+\delta_2)}} e^{-(\delta_1+\delta_2)(r+1)} a^{r+k\beta(r+1,\, k-r)}$$

$$= -\frac{e^{-\delta_1}\left(e^{-(\delta_1+\delta_2)}\right)}{e^{-\delta 1}\left(1 - e^{-\delta_2}\right)} e^{-(\delta_1+\delta_2)(r+1)} a^{r+k\beta(r+1,\, k-r)} \qquad (9)$$

Adding (7), (8) and (9) we get

$$\mu'_r = \left[1 - \frac{e^{-(\delta_1+\delta_2)}}{\left(e^{-\delta_2} - 1\right)} e^{-\delta_1(r+1)} - \frac{e^{-(\delta_1+\delta_2)} e^{-(\delta_1+\delta_2)(r+1)}}{(1 - e^{-\delta_2})}\right] a^{r-k\beta(r+1,\, k-r)} \qquad (10)$$

When r =1, we have

$$\mu'_1 = \left[1 - \frac{e^{-(\delta_1+\delta_2)}}{\left(e^{-\delta_2} - 1\right)} e^{-2\delta_1} - \frac{e^{-(\delta_1+\delta_2)} e^{-2(\delta_1+\delta_2)}}{(1 - e^{-\delta_2})}\right] a^{1-k\beta(2,\, k-1)} \qquad (11)$$

and when r = 2, we have

$$\mu'_2 = \left[1 - \frac{e^{-(\delta_1+\delta_2)}}{\left(e^{-\delta_2} - 1\right)} e^{-3\delta_1} - \frac{e^{-(\delta_1+\delta_2)} e^{-3(\delta_1+\delta_2)}}{(1 - e^{-\delta_2})}\right] a^{2-k\beta(3,k-2)} \qquad (12)$$

2. Data Analysis

The above model is utilised to measure the fertility trend of cohort of 313 households. The data under study was collected while conducting survey by Family Planning Association of India during 1998-99 in suburban parts of Delhi. A circular systematic sampling technique was applied by selecting the first household by using random number table from the random numbers between 1 to 1562 and thereafter every fifth house was picked up until 313 households were selected for collecting the information for all married couples in the house belonging to age 49 and below. Prior to fitting the data into the model, the data were classified according to each parity. Let f_i be the number of females who undergone i^{th} parity and m_i be the number of months awaited by females for i^{th} parity. Further, let λ_i stand for the i^{th} hazard rate.

Then from the data, we have

$f_1 = 316$; $f_2 = 248$; $f_3 = 155$; $m_1 = 8112$; $m_2 = 7226$; $m_3 = 5534$

and $\lambda_1 = 0.467455$; $\lambda_2 = 0.411846$ $\lambda_3 = 0.336104$

$e^{-\delta 1} = \lambda_2 / \lambda_1 = 0.8810378$; $e^{-2\delta 1} = 0.7762277$;

$e^{-3\delta 1} = 0.683886$; $e^{-\delta 2} = \lambda_3 / \lambda_2 = 0.8160914$

$e^{-2\delta 2} = 0.6660052$; $e^{-(\delta 1+\delta 2)} = 0.7190074$

$e^{-2(\delta 1+\delta 2)} = 0.5169717$; $e^{-3(\delta 1+\delta 2)} = 0.3717065$

Substituting in equations (11) and (12), we have

$\mu_1' = 2.0136\ a^{1-k\beta(2,k-1)}$ (13)

$\mu_2' = 2.2205\ a^{2-k\beta(3,k-2)}$ (14)

The computed μ_1' and μ_2' from the data are

$\mu_1' = 8112/316 = 2.13924$ years

$\mu_2' = 7226/248 = 2.42809$ years

Equating these values with equations (13) and (14), we have

$2.13924 = 2.0136\ a^{1-k\beta(2,k-1)}$
$2.42809 = 2.2205\ a^{2-k\beta(3,k-2)}$
Taking $\theta_1 = 1 - k\beta(2, k-1)$
$\theta_2 = 2 - k\beta(3, k-2)$
We have $a^{\theta_1} = 1.624$
$a^{\theta_2} = 1.0935$
$\theta_1 \log a = 0.0263$
$\theta_2 \log a = 0.0388 \Rightarrow \theta_1/\theta_2 = 0.6775$
$1 - k\beta(2,k-1) = 0.6775\,(2 - k\beta(3,k-2))$
which gives $k = 2.8227$
Substituting the value of k in the equations of a^{θ_1}, a^{θ_2} we have
$a^{1-k\beta(2,k-1)} = 1.624$
$a^{2-k\beta(3,k-2)} = 1.0935$
$\Rightarrow a = 3.8229$
The parameters estimated above i.e. a and k are substituted in the model (1) to obtain

$$f_n(s_n)=\frac{(2.8227)\,(3.8229)^{2.8227}}{\Gamma(2.8227)}\sum_{i-1}^{n}\prod_{j\neq i}^{n}\frac{e^{-(\delta_1+\ldots+\delta_{j-1})}e^{-(\delta_1+\ldots\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots+\delta_{j-1})}-e^{-(\delta_1+\ldots+\delta_{i-1})}}\cdot\frac{1}{(3.8229+e^{-(\delta_1+\delta_2+\delta_3\ldots+\delta_{i-1})}s_n)^{3.8227}}$$

So waiting time distribution of s_n for n=1 i.e. s_1 given by

$$f_1(s_1)=\frac{ka^k}{\Gamma(k)}\frac{1}{(a-1.s_n)^{k+1}}$$

Taking into consideration the fact that the hazard rate during first to second parity is changed from λ to $\lambda e^{-\delta i}$, where λ is distributed as

$$\phi(\lambda)=\frac{a^k e^{-a^\lambda}\lambda^{k-1}}{\Gamma(k)}\quad 0<\lambda<\infty \qquad a, k > 0$$

The distribution of s_n for n = 2 (i.e.s_2) is given by

$$f_2(s_2)=\frac{a^k}{\Gamma(k)}\int_0^\infty \lambda e^{-\delta_1}e^{-\lambda_e^{-\delta_1 t}}e^{-\alpha\lambda}d\lambda \qquad =\frac{ka^k e^{-\delta_1}}{(a+e^{-\delta_1})^{k+1}}$$

Similarly the distribution of s_n for n=3, is given by

$$f_3(s_3)=\frac{a^k}{\Gamma(k)}\int_0^\infty \lambda e^{-(\delta_1+\delta_2)}e^{-\lambda_e^{-(\delta_1+\delta_2)t}}e^{-\alpha\lambda}\lambda^{k-1}d\lambda] \qquad =\frac{ka^k e^{-(\delta_1+\delta_2)}}{(a+e^{-(\delta_1+\delta_2)})^{k+1}}$$

Finally the distribution of s_n is given by

$$f_n(s_n)=\frac{a^k}{\Gamma(k)}\int_0^\infty \lambda e^{-(\delta_1+\delta_2+\ldots\ldots+\delta_n)}e^{-\lambda_e^{-(\delta_1+\delta_2+\ldots\ldots+\delta_n)t}}e^{-\alpha\lambda}\lambda^{k-1}d\lambda$$

$$=\frac{ka^k e^{-(\delta_1+\delta_2+\ldots\ldots\delta_n)}}{(a+e^{-(\delta_1+\delta_2+\ldots..+\delta_n)})^{k+1}}$$

Accordingly for s_2, the proportion of women having 2^{nd} birth between x to (x+1) years is given by

$$\int_x^{x+1} f_2(s_2)=a^k\left[\frac{1}{[(a+e^{-\delta_1}x)]^k}-\frac{1}{[(a+e^{-\delta_1}(x+1))]^k}\right]$$

and that between x to x+1 years for s_3 is given by

$$\int_x^{x+1} f_3(s_3)ds_3 = a^k\left[\frac{1}{[(a+e^{-(\delta_1+\delta_2)}x)]^k} - \frac{1}{[(a+e^{--(\delta_1+\delta_2)}(x+1))]^k}\right]$$

Now

$$\int_x^{x+1} f_3(s_3)ds_3 = a^k\left[\frac{1}{[(a+e^{-(\delta_1+\delta_2)}x)]^k} - \frac{1}{[(a+e^{--(\delta_1+\delta_2)}(x+1))]^k}\right]$$

$$\int_x^{x+1} f_1(s_1)ds_1 = a^k\left[\frac{1}{[(a+x)]^k} - \frac{1}{[(a+(x+1))]^k}\right]$$

$$\int_0^1 f_1(s_1)ds_1 = a^k\left[\frac{1}{a^k} - \frac{1}{(a+1)^k}\right] = 0.488206$$

$$\int_1^2 f_1(s_1)ds_1 = a^k\left[\frac{1}{(a+1)^k} - \frac{1}{(a+2)^k}\right] = 0.214488$$

$$\int_2^3 f_1(s_1)ds_1 = a^k\left[\frac{1}{(a+2)^k} - \frac{1}{(a+3)^k}\right] = 0.109032$$

$$\int_3^4 f_1(s_1)ds_1 = a^k\left[\frac{1}{(a+3)^k} - \frac{1}{(a+4)^k}\right] = 0.061344$$

$$\int_x^{x+1} f_2(s_2)ds_2 = a^k\left[\frac{1}{(a+e^{-\delta_1}x)^k} - \frac{1}{(a+e^{-\delta_1}(x+1))^k}\right]$$

$$\int_0^1 f_2(s_2)ds_2 = 0.449997$$

$$\int_1^2 f_2(s_2)ds_2 = 0.214701$$

$$\int_2^3 f_2(s_2)ds_2 = 0.115488$$

$$\int_3^4 f_2(s_2)ds_2 = 0.067712$$

$$\int_x^{x+1} f_3(s_3)ds_3 = a^k\left[\frac{1}{(a+e^{-(\delta_1+\delta_2)}x)^k} - \frac{1}{(a+e^{-(\delta_1+\delta_2)}(x+1))^k}\right]$$

$$\int_0^1 f_3(s_3)ds_3 = 0.39152$$

$$\int_1^2 f_3(s_3)ds_3 = 0.210129$$

$$\int_2^3 f_3(s_3)ds_3 = 0.122998$$

$$\int_3^4 f_3(s_3)ds_3 = 0.07686$$

The fertility for 1st, 2nd and 3rd parity based on the above renewal theory is shown below in the following fertility table.

Time/Parity	**1st**	**2nd**	**3rd**
0-1	0.488206	0.449997	0.391521
1-2	0.214488	0.214701	0.210129
2-3	0.109032	0.115488	0.122998
3-4	0.061344	0.067712	0.07686
4 and above	**0.12693**	**0.152109**	**0.198493**

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