Journal of Reliability and Statistical Studies (ISSN: 0974-8024)
Vol. 1, Issue 1(2008) 25-32

RELIABILITY AND COST ANALYSIS OF A SYSTEM WITH MULTIPLE COMPONENTS USING COPULA

S. B. Pandey [1], S. B. Singh[2] and Shweta Sharma[1]
1. Kumaun University (SSJ Campus), Almora, Uttrakhand, India
2. G. B. Pant University of Agriculture & Technology, Uttrakhand, India
1. sshweta81@yahoo.com, 2. drsurajbsingh@yahoo.com

Abstract

The present study evaluates various reliability measures including transition state probabilities of a sub system on a website which caters to the 'contact us' functionality using copula technique. The system in context uses a combination of both frequently asked questions and contact us forms on the website to help customers getting resolutions to their queries. The failure and repair times for the system follow exponential and general distribution respectively. Introducing two different types of repair between adjacent states the system has been studied to evaluate MTTF, steady state probability, availability and cost analysis by using Gumbel-Houggard family of copula.

Key Words: Reliability, MTTF, Cost Analysis, Copula, Supplementary variable Techniques.

1. Introduction

As the complexity and automation of equipments increased, it resulted in severe problems of maintenance and repair. This put forward the tasks of developing a systematic approach to the study of any phenomena and process that can lead to failure free operation or render service for a good or at least reasonable period of time. Authors [1-7] have described various reliability aspects and it's principles in the modern day life.

The present paper calculates reliability of a sub system on a website which caters to the contact us functionality. Customers get in touch with the enterprise using "Contact us" functionality on the website. Generally there are different channels by which customers try to get the resolution on their queries (frequently asked questions, phone contacts, website contact us form, live chats etc). The system in context is a combination of both frequently asked questions (FAQ) and contact us forms.

2. System Description

These days in the ecommerce market enterprises strive for providing highest quality of customer service to its customers. This is reflected in the design of the website to effective call handling in the call centres. One of the main objectives of this is to provide customers an efficient way of getting their queries resolved. There are different ways in which this can be achieved, like: frequently asked questions (FAQ) section on the website with an intention of capturing the generic queries for the customers, live chats with the call centre agents, contact us form. These "Contact us" forms are designed in a way to capture the key details which will enable back office agents to have enough details to resolve the query. In case the nature of the query is to be in touch with customers, call centre agents do so in the preferred day/time asked by the customers.

The system in context has two main channels namely, frequently asked questions and Contact us forms. Customers come to the website and try to get the queries resolved using FAQ section of the website. In case they do not get answer to the query they will submit a contact us form with details of the query so that customer service agent of the enterprise gets in touch with the customers. The contact us forms are designed in a way to capture customer details in an easy fashion in very less time. One of the facilities which enables this is, users just enters the house number and postcode of the address in which customer lives and system does the address lookup using a "Quick address Lookup" software. This software has a list of all addresses in the country (list is updated regularly to keep it up to date). In case this software is down the customer is not able to lookup his/her address. If the customer is having an overseas address for communication, there is provision of manual address entry.

Keeping the above facts in view and analysing the different possibilities, the system has been mathematically modelled. By incorporating two different types of repair between adjacent states the system has been studied to evaluate MTTF, steady state probability, availability and cost analysis by using Gumbel-Houggard family of copula. The failure and repair times for the system follow exponential and general distribution respectively. Further, the supplementary variable technique is used to study various measures of the system. At last some numerical examples have been taken to observe the particular cases.

3. Assumptions

i. Initially the system is in good state.
ii. System is partly functional if frequently asked questions repository is down as contact us functionality is available as an alternative.
iii. Contact us form uses quick address search functionality which enables users to find the address where they are based instead of manually entering it.
iv. System is assumed to be functional even if quick address search is not working; this is because customers can manually enter the address on the website.
v. In case database is down, system is not able to store the information entered by the customers on the contact us forms and is assumed to be completely failed.
vi. The repair of a failed unit starts at once.
vii. The repaired unit works like a new one.
viii. The failure and repair time for the system follows exponential and general distribution respectively.
ix. Transition from the completely failed state S_3 to the initial state S_0 follows two different distributions.
x. Joint probability distribution of repair rate from completely failed state S_3 to the initial state S_0 follows Gumbel- Houggard family of Copula.

4. State Transition Diagram

Figure 1 represents the state transition diagram of the system.

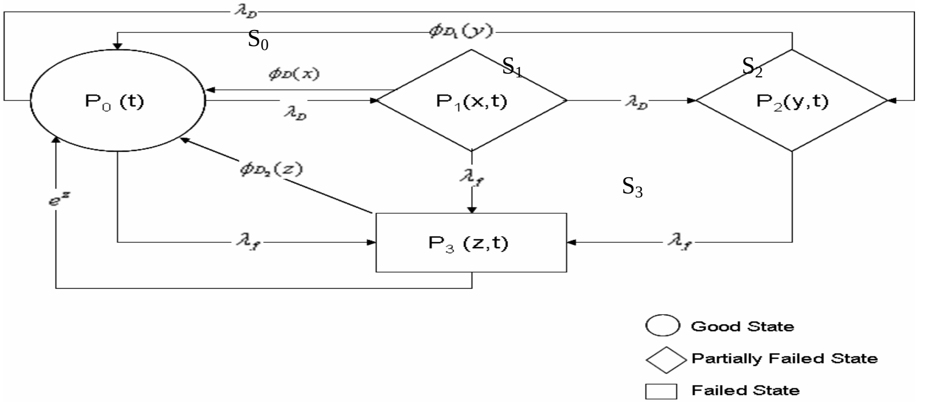

Figure 1: State transition diagram

5. Notations

S_0 : Denotes the state when all the operating units are in working condition.
S_1 : Denotes the state when quick address search has failed.
S_2 : Denotes the state when frequently asked questions section on the website has failed.

S_3: Denotes the state when the database has failed and the system is in completely failed state.

λ_D, λ_f : Constant failure rates of the main/standby units.

$\phi_D(x), S_D(x)$: Repair rate and probability density function of the system state S_1 in elapsed repair time x.

$\phi_{D_1}(y), S_{D_1}(y)$: Repair rate and probability density function of the system state S_2 in elapsed repair time y.

$\phi_{D_2}(z), S_{D_2}(z)$: Repair rate and probability density function of the system state S_3 in elapsed repair time z.

$P_i(t)$: Probability that the system is in S_i state at instant $'t'$ for $i = 0$ to 3.

$\overline{P}(s)$: Laplace transformation of P (t).

$F_Z(z)/F$: Marginal distribution of random variables, where $F_Z(z) = e^Z$ and $F = \phi_{D_2}(z)$.

Letting $F_Z(z) = e^Z$ and $F = \phi_{D_2}(z)$, the expression for joint probability according to Gumbel-Houggard family is given by

$$\exp[z^\theta + \log(\phi_{D_2}(z))^\theta]^{1/\theta}.$$

6. Formulation of Mathematical Model

By probability of considerations and continuity arguments we can obtain the following set of difference - differential equations governing the present mathematical model

$$\left[\frac{\partial}{\partial t} + 2\lambda_D + \lambda_f\right]P_0(t) = \sum_{i=1}^{i=2}\int_0^\infty P_i(\varphi,t)\phi_\alpha(\varphi)d\varphi + \int_0^\infty P_3(z,t)\exp\{z^\theta + \{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}dz \tag{1}$$

Where

$$\begin{bmatrix} i = 1, \varphi = x, \alpha = D \\ i = 2, \varphi = y, \alpha = D_1 \end{bmatrix}$$

$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \lambda_D + \lambda_f + \phi_D(x)\right]P_1(x,t) = 0 \tag{2}$$

$$\left[\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \lambda_f + \phi_{D_1}(y)\right]P_2(y,t) = 0 \tag{3}$$

$$\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial t} + \exp\{z^\theta + \{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}\right]P_3(z,t) = 0 \tag{4}$$

The boundary conditions are:

$$P_1(0,t) = \lambda_D P_0(t) \tag{5}$$

$$P_2(0,t) = \lambda_D P_1(t) + \lambda_D P_0(t) \tag{6}$$

$$P_3(0,t) = \lambda_f[P_0(t) + P_1(t) + P_2(t)] \tag{7}$$

Initial conditions are:

$P_0(0) = 1$, and other state probabilities are zero at time t = 0. (8)

7. Solution of the model

Taking Laplace transforms of equations (1)-(7) and using equation (8), one gets equations (9)-(12).

$$[s+2\lambda_D+\lambda_f]\overline{P_0}(s)=1+\sum_{i=1}^{i=2}\int_0^{\infty}\overline{P_i}(\varphi,s)\phi_\alpha(\varphi)d\varphi+\int_0^{\infty}\overline{P_3}(z,s)\exp\{z^\theta+\{\log\phi_\alpha(z)\}^\theta\}^{1/\theta}d\theta \tag{9}$$

$$\begin{bmatrix} i=1,\theta=x,\alpha=D \\ i=2,\theta=y,\alpha=D_1 \end{bmatrix}$$

$$[\frac{\partial}{\partial x}+s+\lambda_D+\lambda_f+\phi_D(x)]\overline{P_1}(x,s)=0 \tag{10}$$

$$[\frac{\partial}{\partial y}+s+\lambda_f+\phi_{D_1}(y)]\overline{P_2}(y,s)=0 \tag{11}$$

$$[\frac{\partial}{\partial z}+s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}]\overline{P_3}(z,s)=0 \tag{12}$$

Boundary conditions are:

$$\overline{P_1}(0,s)=\lambda_D\overline{P_0}(s) \tag{13}$$

$$\overline{P_2}(0,s)=\lambda_D\overline{P_1}(s)+\lambda_D\overline{P_0}(s) \tag{14}$$

$$\overline{P_3}(0,s)=\lambda_f[\overline{P_0}(s)+\overline{P_1}(s)+\overline{P_2}(s)] \tag{15}$$

Solving equations (9) - (12) and using equations (13) - (15), one can get equations (16)-(19)

$$\overline{P_0}(s)=1/\{s[1+\lambda_D r_D(s+\lambda_D+\lambda_f)+\lambda_D^2 r_D(s+\lambda_D+\lambda_f)r_{D_1}(s+\lambda_f)+\lambda_D r_{D_1}(s+\lambda_f)][1+\lambda_f r_{D_2}(s)]\} \tag{16}$$

$$\overline{P_1}(s)=\lambda_D r_D(s+\lambda_D+\lambda_f)/\{s[1+\lambda_D r_D(s+\lambda_D+\lambda_f)+\lambda_D^2 r_D(s+\lambda_D+\lambda_f)r_{D_1}(s+\lambda_f)+\lambda_D r_{D_1}(s+\lambda_f)][1+\lambda_f r_{D_2}(s)]\} \tag{17}$$

$$\overline{P_2}(s)=\lambda_D r_{D_1}(s+\lambda_f)[\lambda_D r_D(s+\lambda_D+\lambda_f)+1]/\{s[1+\lambda_D r_D(s+\lambda_D+\lambda_f)+\lambda_D^2 r_D(s+\lambda_D+\lambda_f)r_{D_1}(s+\lambda_f)+\lambda_D r_{D_1}(s+\lambda_f)][1+\lambda_f r_{D_2}(s)]\} \tag{18}$$

$$\overline{P_3}(s)=\lambda_f r_{D_2}(s)/\{s[1+\lambda_f r_{D_2}(s)]\} \tag{19}$$

where,

$$r_i(s)=\frac{1-\overline{S_i}(s)}{s}$$

The Laplace transforms of the probabilities that the system is in up (i.e. good or partially failed) and failed states at time‘t’, are as follows

$$\overline{P_{up}}(s)=\sum_{i=0}^{2}\overline{P_i}(s)$$

$$\overline{P_{up}}(s)=\{s[1+\lambda_f r_{D_2}(s)]\}^{-1} \tag{20}$$

$$\overline{P}_{failed}(s) = \overline{P}_3(s) \tag{21}$$

Also it is interesting to note that

$$\overline{P}_{up}(s) + \overline{P}_{failed}(s) = 1/s$$

8. Asymptotic behavior of the system

Using Abel's lemma $\lim\limits_{s\to 0}[s\overline{F}(s)] = \lim\limits_{t\to\infty} A(t) = F$ (say) in equations (4.16)-(4.21), provided the limit on the right hand exists, the following time independent probabilities are obtained

$$P_0 = \frac{1}{[1+\lambda_D r_D(\lambda_D+\lambda_f)+\lambda_D{}^2 r_D(\lambda_D+\lambda_f)r_{D_1}(\lambda_f)+\lambda_D r_{D_1}(\lambda_f)][1+\lambda_f r_{D_2}(0)]}$$

$$P_1 = \frac{\lambda_D r_D(\lambda_D+\lambda_f)}{[1+\lambda_D r_D(\lambda_D+\lambda_f)+\lambda_D{}^2 r_D(\lambda_D+\lambda_f)r_{D_1}(\lambda_f)+\lambda_D r_{D_1}(\lambda_f)][1+\lambda_f r_{D_2}(0)]}$$

$$P_2 = \frac{\lambda_D r_{D_1}(\lambda_f)[\lambda_D r_D(\lambda_D+\lambda_f)+1]}{[1+\lambda_D r_D(\lambda_D+\lambda_f)+\lambda_D{}^2 r_D(\lambda_D+\lambda_f)r_{D_1}(\lambda_f)+\lambda_D r_{D_1}(\lambda_f)][1+\lambda_f r_{D_2}(0)]}$$

$$P_3 = \frac{\lambda_f r_{D_2}(0)}{[1+\lambda_f r_{D_2}(0)]}$$

$$P_{up} = \frac{1}{[1+\lambda_f r_{D_2}(0)]}$$

$$P_{failed} = P_3$$

9. Particular Case

When repair follows exponential time distribution, setting,

$$\overline{S_D}(s) = \frac{\phi_D}{s+\phi_D},\ \overline{S_{D_1}}(s) = \frac{\phi_{D_1}}{s+\phi_{D_1}} \text{ and } \overline{S_{D_2}}(s) = \frac{\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}}{s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}}.$$

equations (16)-(21) yields,

$$\overline{P}_0(s) = 1/\{s[1+\frac{\lambda_D}{(s+\lambda_D+\lambda_f+\phi_D)}+\frac{\lambda_D{}^2}{(s+\lambda_D+\lambda_f+\phi_D)(s+\lambda_f+\phi_{D_1})}+\frac{\lambda_D}{(s+\lambda_f+\phi_{D_1})}]$$
$$[1+\frac{\lambda_f}{(s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta})}]\} \tag{22}$$

$$\overline{P}_1(s) = \lambda_D/\{s[1+\frac{\lambda_D}{(s+\lambda_D+\lambda_f+\phi_D)}+\frac{\lambda_D{}^2}{(s+\lambda_D+\lambda_f+\phi_D)(s+\lambda_f+\phi_{D_1})}+\frac{\lambda_D}{(s+\lambda_f+\phi_{D_1})}]$$
$$[1+\frac{\lambda_f}{(s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta})}]\} \tag{23}$$

$$\overline{P_2}(s) = \lambda_D[\frac{\lambda_D}{(s+\lambda_D+\lambda_f+\phi_D)(s+\lambda_f+\phi_{D_1})} + \frac{1}{(s+\lambda_f+\phi_{D_1})}]/\{s[1+\frac{\lambda_D}{(s+\lambda_D+\lambda_f+\phi_D)}$$
$$+\frac{\lambda_D{}^2}{(s+\lambda_D+\lambda_f+\phi_D)(s+\lambda_f+\phi_{D_1})} + \frac{\lambda_D}{(s+\lambda_f+\phi_{D_1})}]$$
$$[1+\frac{\lambda_f}{(s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta})}]\} \tag{24}$$

$$\overline{P_3}(s) = \frac{\lambda_f}{s[s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}+\lambda_f]} \tag{25}$$

$$\overline{P_{up}}(s) = \frac{s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}}{s[s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}+\lambda_f]} \tag{26}$$

$$\overline{P}_{failed}(s) = \overline{P_3}(s) \tag{27}$$

10. Numerical Computations

10.1. Availability Analysis

Setting the numerical values say:

$$\lambda_D = \lambda_f = 0.5,\ \phi_D = \phi_{D_1} = \phi_{D_2} = 1$$

and

$$\overline{S_D}(s) = \frac{\phi_D}{s+\phi_D},\ \overline{S_{D_1}}(s) = \frac{\phi_{D_1}}{s+\phi_{D_1}},\ \overline{S_{D_2}}(s) = \frac{\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}}{s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}},$$

in equation (20) and then taking Inverse Laplace transform, we get

$$P_{up}(t) = 0.84464 + 0.15536e^{-3.21828t}$$

$$P_{failed}(t) = 0.15536 - 0.15536e^{-3.21828t}$$

The values of P_{up} (t) and P_{failed} (t) for different values of 't' is shown in Table 1 and the corresponding graph is shown in the Figure 2.

S. No.	Time t	$P_{up}(t)$	P_{failed} (t)
1	0	1	0
2	1	0.85086	0.14914
3	2	0.84489	0.15511
4	3	0.84465	0.15535
5	4	0.84464	0.15536
6	5	0.84464	0.15536
7	6	0.84464	0.15536
8	7	0.84464	0.15536
9	8	0.84464	0.15536
10	9	0.84464	0.15536
11	10	0.84464	0.15536

Table 1: System reliability data

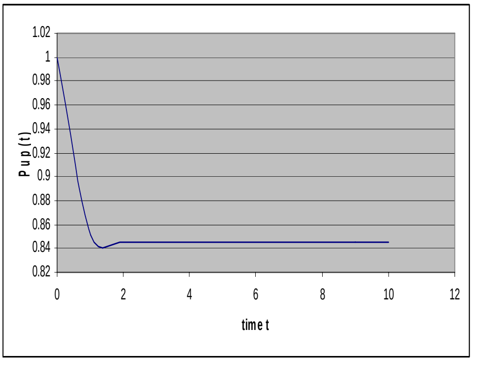

Figure 2: Graph of reliability Vs time

10.2. Mean Time To Failure (MTTF)

Substituting $\overline{S_i}(s)=0$ where $r_i(s)=\frac{1-\overline{S_i}(s)}{s}$ in equation (20), we can obtain

$$M.T.T.F = \lim_{s\to 0} \overline{P}_{up}(s)$$
$$= \frac{1}{\lambda_f}$$

Setting the numerical values, say

$\lambda_f = 0.01, 0.02, 0.03, \ldots$ one can get the Table 2 and the Figure 3.

S.No.	Failure rate	MTTF
1	0.01	100.00000
2	0.02	50.00000
3	0.03	33.33333
4	0.04	25.00000
5	0.05	20.00000
6	0.06	16.66667
7	0.07	14.28571
8	0.08	12.50000
9	0.09	11.11111
10	0.10	10.00000

Table 2: Mean Time To Failure

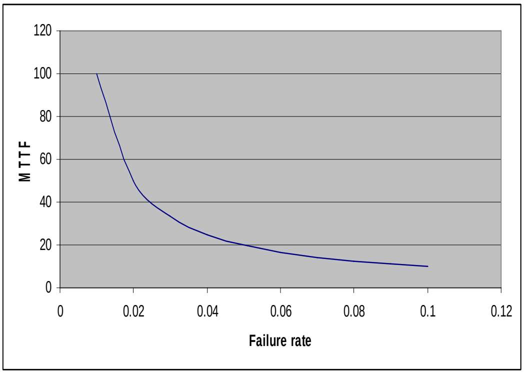

Figure 3: Graph of MTTF Vs Failure rate

10.3. Cost Analysis

Setting the numerical values say

$\lambda_D = \lambda_f = 0.5$, $\phi_D = \phi_{D_1} = \phi_{D_2} = 1$

and

$$\overline{S_D}(s)=\frac{\phi_D}{s+\phi_D},\ \overline{S_{D_1}}(s)=\frac{\phi_{D_1}}{s+\phi_{D_1}},\ \overline{S_{D_2}}(s)=\frac{\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}}{s+\exp\{z^\theta+\{\log\phi_{D_2}(z)\}^\theta\}^{1/\theta}},$$

in equation (18) and then taking Inverse Laplace transform, we get

$$P_2(t)=0.21116t+0.01981e^{-3.21828t}+0.0737e^{-2t}-0.09351$$

Let 'M' and 'N' be the revenue per unit time and service cost per unit time respectively, then the expected profit E(t) during the interval]0,t] is given by

$$E(t)=M\int_0^t P_{up}(t)dt - N[t-\int_0^t P_2(t)dt]$$
$$= M[0.84464t-0.04827e^{-3.21828t}+0.04827]$$
$$-N[0.78884t-0.01981e^{-3.21828t}-0.0737e^{-2t}+0.09351] \quad (28)$$

Setting M = 1 and N = 0.05, 0.1, 0.5 then equation (28) yields Table 3 and the corresponding graph is shown in Figure 4.

S.No.	Time t	Expected profit E(t)		
		N=0.05	N=0.1	N=0.5
1	0	0	0	0
2	1	0.84740	0.80382	0.45519
3	2	1.65399	1.57049	0.90257
4	3	2.45919	2.33620	1.35226
5	4	3.26439	3.10194	1.80241
6	5	4.06958	3.86770	2.25262
7	6	4.87478	4.63345	2.70283
8	7	5.67998	5.39921	3.15305
9	8	6.48518	6.16497	3.60327
10	9	7.29038	6.93072	4.05349
11	10	8.09557	7.69648	4.50371

Table 3: Cost Analysis of the system

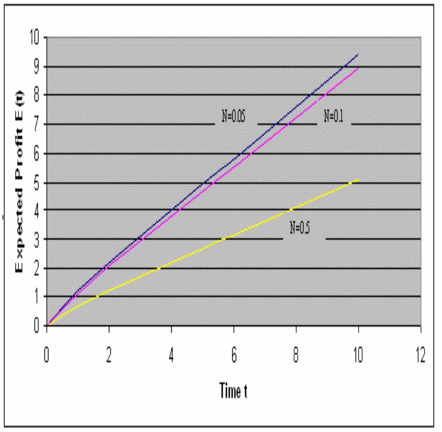

Figure 4: Expected profit Vs time

11. Interpretation of Result

The system reliability obtained in Table (1) shows that the reliability decreases with time but in the long run it approaches to a constant value 0.84464. From Table (2) we observe that M.T.T.F. decreases rapidly in the beginning, but later on with the passage of time, it decreases approximately at a uniform rate. Further, by analyzing the graph shown in Figure (4) we can conclude that as the service cost N approaches towards 1 from N = 0.1, the relative expected profit decreases but below N = 0.1, this decrease is very low.

Acknowledgement

Authors are thankful to the referees whose valuable comments helped in improving the quality of the paper.

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