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BAYES ESTIMATORS OF THE SCALE PARAMETER OF A GENERALIZED GAMMA TYPE MODEL

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Abstract

Bayes estimators of the scale parameter of a generalized gamma type model are obtained for different priors. For the proposed prior, Bayes estimator of θ for given p and k=1 coincides with MLE of θ if c=2, whereas Bayes estimator of hazard rate function of θ given p and k coincides with its MLE. Thumb's rule has been used for constructing a conjugate prior for θ. Bayes estimators of reliability function and hazard rate function have also been obtained.

Key Words: Lifetime Model, Bayes estimator, Maximum Likelihood Estimator, Prior, Reliability Function, Hazard Rate Function.

1. Introduction

In lifetime distributions, Bayesian analysis plays a very important role but its implementation is so tough because if one attempts to implement Bayesian analysis using lifetime models, the likelihood function and the prior provide quite intractable posterior forms which are impossible to analyze analytically and are even very challenging from the conventional numerical perspective. Stacy (1962) proposed a new generalized gamma model and gave its characteristics and applications to life testing. Stacy and Mihram (1965) derived estimators of the generalized gamma distribution. El-Sayyed (1967) derived some new estimators which were unbiased with respect to some loss functions for the parameters in exponential distribution. Upadhyay et al. (2000) used Monte Carlo simulation technique for the Bayesian computation in life testing and reliability models. Pandey and Rao (2006) derived Bayes estimators of the scale parameter of generalized gamma distribution by taking quasi, inverted gamma and uniform prior distributions using precautionary loss function.

In this paper, we have obtained conjugate prior for the proposed model by applying Thumb rule and Bayes estimators of the scale parameter θ of the proposed model under different priors viz. uniform, Jeffrey's, exponential, Mukherjee-Islam, Weibull, gamma etc. by using Lindley's (1980) approach. Bayes estimators of reliability and hazard rate functions under different priors have also been obtained.

The probability density function f(t) of the proposed generalized gamma type model is

$$f(t) = \frac{p}{\theta^k \Gamma k} t^{pk-1} e^{\left\{\frac{-t^p}{\theta}\right\}} I_{(0,\infty)}(t) \qquad p>0,\ \theta>0,\ k>0 \tag{1}$$

Then Likelihood function (L) of (1) is given by

$$L = \left[\frac{p}{\theta^k \Gamma k}\right]^n e^{-\sum_{i=1}^{n} \frac{t_i^p}{\theta}} \prod_{i=1}^{n} t_i^{pk-1} \tag{2}$$

The posterior distribution of θ given the random sample when p and k are fixed is

$$f\left(\theta \middle| \underset{\sim}{t}\right) = \frac{\frac{1}{\theta^{nk}} e^{-\sum_{i=1}^{n} t_i^{p}/\theta} g(\theta)}{\int_{\theta} \frac{1}{\theta^{nk}} e^{-\sum_{i=1}^{n} t_i^{p}/\theta} g(\theta) d\theta} \tag{3}$$

Where g (θ) is the prior of θ. A distinctive feature of the Bayesian approach is the introduction of a prior density to represent prior information about the possible values of the parameters of the model. There are three distinct Bayesian approaches for selection of prior distribution [Diaconis and Ylvisker (1985)]. The choice of a convenient prior distribution which combines easily with the likelihood function has recently been simplified by the construction of conjugate family. The concept of conjugate family was introduced first by Bernard (1954) and fully explained by Raiffa and Schlaifer (1961). The restriction to the conjugate family is not necessary, but it has the advantage that the posterior distribution belongs to the same family.

2. Thumb Rule for Constructing a Conjugate Prior

We have used the concept of kernel (see Raiffa and Schlaifer (1961)) to construct a conjugate prior and the procedure is as given below :

If the probability density function of θ is g, where g denotes either prior or posterior density, and if m is another function on θ such that $g(\theta) = m(\theta) \Big/ \int m(\theta) d\theta$, that is, the ratio $m(\theta)/g(\theta)$ is a constant as regards θ, then we write $g(\theta) \propto m(\theta)$ and say m is a kernel of the density of θ (provided $\int m(\theta) d\theta$ is finite).

Suppose $\alpha(\underset{\sim}{t})$ is a sufficient statistic for the parameter of θ, so that likelihood function $L(\theta|\underset{\sim}{t})$ is,

$$L(\theta|\underset{\sim}{t}) = m[\alpha(\underset{\sim}{t})|\theta] h[\underset{\sim}{t}]$$

where $m[\alpha(\underset{\sim}{t})|\theta]$ is a kernel of the likelihood function. Replace all the terms in the kernel of the likelihood function that are the functions of the sample by prior hyperparameters say $\underset{\sim}{a} = (a_1, a_2, \_\_\_\_\_ \, a_n)$. Then the conjugate prior is $g(\theta) \propto m[\alpha(t)|\theta]_{\alpha(\underset{\sim}{t}) = \underset{\sim}{a}}$

The prior distribution g (θ) may involve some unknown parameter(s) and in order to distinguish them with parameters of the sampling distribution $f(t|\theta)$, we call the parameters of the prior as hyperparameters.

In the case of proposed model, suppose $T_1, T_2, \_\_\_\_ , T_n$ is a random sample from (1) with p>0 and k>0 (both are known) and its likelihood function is given by (2). Since $\alpha = \sum_{i=1}^{n} T_i^{\,p}$ is a sufficient statistic for θ, a kernel of the likelihood function is

$$m[\alpha|\theta] = \frac{e^{-\alpha/\theta}}{\theta^{nk}}.$$

Therefore, the conjugate prior is $g(\theta) \propto \dfrac{e^{-a/\theta}}{\theta^{c}}$, which is a inverted gamma $(c-1, a)$ with hyperparameters a and c .Then from (3) the posterior distribution of θ is

$$f\left(\theta\middle|\underset{\sim}{t}\right)=\frac{\frac{1}{\theta^{nk+c}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,e^{-a/\theta}}{\int_{\theta}\frac{1}{\theta^{nk+c}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,e^{-a/\theta}d\theta}=R\,\frac{e^{-\left(\sum_{i=1}^{n}t_i^{p}+a\right)/\theta}}{\theta^{nk+c}} \tag{4}$$

where $R^{-1}=\int_{\theta}\frac{e^{-\left(\sum_{i=1}^{n}t_i^{p}+a\right)/\theta}}{\theta^{nk+c}}d\theta$

ande Bayes estimator [$w(\theta)^{B}$] of $w(\theta)$ is

$$w(\theta)^{B}=E[w(\theta/\underset{\sim}{t})]=\int w(\theta)f\left(\theta\middle|\underset{\sim}{t}\right)d\theta=\frac{\int_{\theta}w(\theta)\frac{1}{\theta^{nk+c}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,e^{-a/\theta}d\theta}{\int_{\theta}\frac{1}{\theta^{nk+c}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,e^{-a/\theta}d\theta} \tag{5}$$

If $w(\theta)=\theta$, Then

$$\theta^{B}=E[w(\theta/\underset{\sim}{t})]=\int\theta f\left(\theta\middle|\underset{\sim}{t}\right)d\theta=\frac{\int_{0}^{\infty}\theta\frac{1}{\theta^{nk+c}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,e^{-a/\theta}d\theta}{\int_{0}^{\infty}\frac{1}{\theta^{nk+c}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,e^{-a/\theta}d\theta}$$

$$=\frac{\sum_{i=}^{n}t_i^{p}+a}{nk+c-2}\,;\,[\,c\geq 2\,] \tag{6}$$

If c = 2 and a = 0 then Bayes estimator of θ coincides with MLE of θ.

Now we use an improper prior $g(\theta)\propto\theta^{-1}$

The posterior distribution of θ given the random sample when p and k are fixed, is given by (3).

Hence
$$f\left(\theta\middle|\underset{\sim}{t}\right)=\frac{\frac{1}{\theta^{nk}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,\frac{1}{\theta}}{\int_{\theta}\frac{1}{\theta^{nk}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}\,\frac{1}{\theta}d\theta}=\frac{\frac{1}{\theta^{nk+1}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}}{\int_{\theta}\frac{1}{\theta^{nk+1}}e^{-\sum_{i=1}^{n}t_i^{p}/\theta}d\theta}$$

And Bayes estimator (θ^{B}) of θ is given by

$$E[\theta/\underset{\sim}{t})]=\int_{\theta}\theta f\left(\theta\middle|\underset{\sim}{t}\right)d\theta=\frac{\sum_{i=}^{n}t_i^{p}}{nk-1} \tag{7}$$

3. Lindley’s Approach

Bayes estimators are often obtained as the ratio of two integrals which can not be solved by using asymptotic expansion and calculus of difference. Lindley (1980) developed an asymptotic approximation to the ratio

$$I = \frac{\int_{\beta} u(\beta) e^{L(\beta)} g(\beta) d\beta}{\int_{\beta} e^{L(\beta)} g(\beta) d\beta} \tag{8}$$

where $L(\beta)$ is the logarithm of the likelihood function.
According to him

$$1 \approx u(\beta^*) + \frac{\sigma^2}{2}\left[u_2(\beta^*) + 2u_1(\beta^*) p_1(\beta^*)\right] + \frac{\sigma^4}{2}\left[L_3(\beta^*) u_1(\beta^*)\right] \tag{9}$$

where β^* is the MLE of β.

Also, $$L_k(\beta^*) = \frac{\partial^k}{\partial \beta^k} L(\beta)\Big|_{\beta=\beta^*} \tag{10}$$

$$u_k(\beta^*) = \frac{\partial^k}{\partial \beta^k} u(\beta)\Big|_{\beta=\beta^*} \tag{11}$$

$$\sigma^2 = -L_2^{-1}(\beta^*) \tag{12}$$

$$p_k(\beta^*) = \frac{\partial^k}{\partial \beta^k} \log_e g(\beta)\Big|_{\beta=\beta^*} \tag{13}$$

For the model (1), when p and k are fixed; likelihood function is given as

$$L_3(\theta^*) = \frac{-2nk}{\theta^{*3}} + \frac{6\sum t_i^{\,p}}{\theta^{*4}} = \frac{-2nk}{\theta^{*3}} + \frac{6nk\theta^*}{\theta^{*4}} = \frac{4nk}{\theta^{*3}} \qquad [\because \theta^* = \sum_{i=1}^{n} t_i^{\,p} / nk\ ]$$

4. Bayes estimator of θ given p and k

$$u(\theta) = \theta,\ u(\theta^*) = \theta^*,\ u_1(\theta^*) = 1,\ u_2(\theta^*) = 0,\ \sigma^2 = \theta^{*2}/nk,\ \sigma^4 = \theta^{*4}/n^2k^2 \tag{14}$$

Bayes estimators **(** θ^B **)** of θ given p and k under different priors are given in Table-1, along with the values of $p_1(\theta^*)$.

If c=2, Bayes estimator of θ (given p and k) for the proposed prior coincides with MLE of θ**.**

5. Bayes estimator of Reliability Function

Reliability Function of the given model is given by

$$R(t) = \int_t^\infty f(u)du = \frac{p}{\theta^k \Gamma k} \int_t^\infty u^{pk-1} e^{-\frac{u^p}{\theta}} du \qquad [\ u^p/\theta = y\ ]$$

$$= \frac{1}{\Gamma k} \int_{t^p/\theta}^{\infty} e^{-y} y^{k-1} dy \tag{15}$$

the integral of (15) may be evaluated exactly when k=1, in which case, we have

$$R(t) = e^{-t^p/\theta} \tag{16}$$

Bayes estimators **(** $R(t)_\theta^{\,B}$ **)** of Reliability Function given p for k=1 by using different priors are given in Table-2.

6. Bayes estimator of Hazard Rate Function:

Hazard Rate Function H(t) may be given as

$$\text{H (t)} = \frac{f(t)}{R(t)} = \frac{\frac{p}{\theta^{k}\Gamma k} t^{pk-1} e^{-\frac{t^p}{\theta}} I_{(0,\infty)}(t)}{\frac{1}{\Gamma k}\int_{t^p/\theta}^{\infty} e^{-y} y^{k-1} dy} \tag{17}$$

Special Case When k=1, $\quad \text{H (t)} = \frac{f(t)}{R(t)} = \frac{p}{\theta} t^{p-1}$ (18)

Then $\quad u(\theta) = \frac{p}{\theta} t^{p-1}, \qquad u(\theta^*) = \frac{p}{\theta^*} t^{p-1},$

$$u_1(\theta^*) = \frac{-pt^{p-1}}{\theta^{*2}}, \qquad u_2(\theta^*) = \frac{2p}{\theta^{*3}} t^{p-1} \tag{19}$$

Bayes estimator **(** $H(t)_{\theta}^{B}$ **)** of H(t) given p for k=1 by using different priors are given in Table-3. It is easy to see that the Bayes estimator of H(t) coincides with its MLE under Jeffrey's prior whereas under uniform and Mukherjee-Islam priors, it is better than its MLE.

Prior	**Density**	$p_1(\theta^*)$	θ^B
Uniform	$I_{(0,1)}(\theta)$	0	$\left[1+\frac{2}{nk}\right]\theta^*$
Jeffrey's	$\theta^{-1} I_{(1,e)}(\theta)$	$-\frac{1}{\theta^*}$	$\left[1+\frac{1}{nk}\right]\theta^*$
Exponential	$e^{-\theta}$; $\theta > 0$	-1	$\left[1+\frac{2}{nk}\right]\theta^* - \frac{\theta^{*2}}{nk}$
Mukherjee-Islam	$\alpha\sigma^{-\alpha}\theta^{\alpha-1} I_{(0,\sigma)}(\theta)$; $\sigma,\alpha > 0$	$\frac{\alpha-1}{\theta^*}$	$\left[1+\frac{\alpha+1}{nk}\right]\theta^*$
Weibull	$\frac{\alpha}{\sigma}\theta^{\alpha-1} e^{-\theta^{\alpha}/\sigma}$; $\alpha,\sigma > 0, \theta > 0$	$\frac{\alpha-1}{\theta^*} - \frac{\alpha}{\sigma}\theta^{*\alpha-1}$	$\left[1+\frac{\alpha+1}{nk}\right]\theta^* - \frac{\alpha}{nk\sigma}\theta^{*\alpha+1}$
Gamma	$\frac{1}{\sigma^{\alpha}\Gamma\alpha}\theta^{\alpha-1} e^{-\theta/\sigma}$; $\alpha,\sigma > 0, \theta > 0$	$\frac{\alpha-1}{\theta^*} - \frac{1}{\sigma}$	$\left[1+\frac{\alpha+1}{nk}\right]\theta^* - \frac{\theta^{*2}}{nk\sigma}$
Proposed	$\frac{c-1}{\theta^c} I_{(1,\infty)}(\theta)$	$\frac{-c}{\theta^*}$	$\left[1-\frac{c-2}{nk}\right]\theta^*$
Generalized Gamma	$\frac{p}{\sigma^k\Gamma k}\theta^{pk-1} e^{-\theta^p/\sigma}$. $I_{(0,\infty)}(\theta)$	$\frac{pk-1}{\theta^*} - \frac{p\theta^{*p-1}}{\sigma}$	$\left[1+\frac{pk+1}{nk}\right]\theta^* - \frac{p\theta^{*p+1}}{nk\sigma}$

Table-1: Bayes estimators (θ^B **) of** θ **given p and k under different priors**

7. Illustration

A random sample of size 25 is generated from the proposed model with k = 1, p = 2 and θ = 4. Table 4 shows the Bayes estimates of θ for p = 2, k = 1 and corresponding Bayes estimates of reliability and hazard rate functions. Table-4 reveals that Bayes estimator of θ for p = 2, k = 1 is quite close to its true value under gamma prior, as well as Bayes estimators of k for θ = 4, p = 2 also seems to be closer to its true value under gamma prior. The calculations for reliability and hazard rate functions may also be performed in a similar manner at different values of t considering different priors.

Priors	$R(t)_\theta^{\ B}$
Uniform	$R^*(t)\left[1+\frac{t^p}{n\theta*}\left(\frac{t^p}{2\theta*}+1\right)\right]$
Jeffrey's	$R^*(t)\left[1+\frac{t^{2p}}{2n\theta*^2}\right]$
Exponential	$R^*(t)\left[1+\frac{t^p}{2n\theta*^2}\left(t^p+2\theta*(1-\theta*)\right)\right]$
Mukherjee-Islam	$R^*(t)\left[1+\frac{t^p}{2n\theta*^2}\left(t^p+2\alpha\theta*\right)\right]$
Weibull	$R^*(t)\left[1+\frac{t^p}{2n\theta*^2}\left(t^p+2\alpha\theta*(1-\theta*^\alpha/\sigma\right)\right]$
Gamma	$R^*(t)\left[1+\frac{t^p}{2n\theta*^2}\left(t^p+2\theta*(\alpha-\sigma^{-1})\right)\right]$
Proposed	$R^*(t)\left[1+\frac{t^p}{2n\theta*^2}\left(t^p+2\theta*(1-c)\right)\right]$
Generalized Gamma	$R^*(t)\left[1+\frac{t^p}{2n\theta*^2}\left(t^p+2\theta*pk-2p\theta*^{p+1}/\sigma\right)\right]$

Table-2: Bayes estimators ($R(t)_\theta^{\ B}$) of Reliability Function given p for k=1

Priors	$H(t)_\theta^{\ B}$
Uniform	$H^*(t)\left[1-\frac{1}{n}\right]$
Jeffrey's	$H^*(t)$
Exponential	$H^*(t)\left[1-\frac{1-\theta*}{n}\right]$
Mukherjee-Islam	$H^*(t)\left[1-\frac{\alpha}{n}\right]$
Weibull	$H^*(t)\left[1-\frac{\alpha}{n}\left(1-\theta*^\alpha/\sigma\right)\right]$
Gamma	$H^*(t)\left[1-\frac{1}{n}\left(\alpha-1/\sigma\right)\right]$
Proposed	$H^*(t)\left[1+\frac{c-1}{n}\right]$
Generalized Gamma	$H^*(t)\left[1-\frac{pk}{n}+\frac{p\theta*^p}{n\sigma}\right]$

Table-3: Bayes estimator ($H(t)_\theta^{\ B}$) of Hazard Rate Function given p for k=1

Prior	θ^B	$R(t)^B$					$H(t)^B$				
		t =0	**1**	**2**	**3**	**4**	**t = 0**	**1**	**2**	**3**	**4**
Uniform	3.4671	1.00	0.75	0.32	0.08	0.012	0.00	0.58	1.17	1.75	2.34
Jeffrey's	3.3387	1.00	0.74	0.30	0.06	0.007	0.00	0.61	1.22	1.83	2.44
Exponential	3.0549	1.00	0.72	0.27	0.06	0.007	0.00	0.66	1.33	1.99	2.66
Mukh.-Islam											
$\alpha = 1$	3.4671	1.00	0.75	0.32	0.08	0.012	0.00	0.58	1.17	1.75	2.34
$= 2$	3.5955	1.00	0.76	0.33	0.08	0.014	0.00	0.54	1.08	1.61	2.15
$= 3$	3.7239	1.00	0.76	0.34	0.09	0.015	0.00	0.53	1.07	1.60	2.14
Weibull											
$\alpha = 1, \sigma = 1$	3.0549	1.00	0.72	0.27	0.06	0.007	0.00	0.66	1.33	1.99	2.66
$= 2$	3.2610	1.00	0.73	0.29	0.07	0.010	0.00	0.61	1.22	1.83	2.45
$= 3$	3.3297	1.00	0.73	0.30	0.07	0.011	0.00	0.60	1.20	1.80	2.40
Gamma											
$\alpha = 1, \sigma = 1$	3.0549	1.00	0.74	0.30	0.07	0.011	0.00	0.61	1.22	1.83	2.45
$= 2$	3.2610	1.00	0.74	0.31	0.08	0.012	0.00	0.60	1.99	1.80	2.40
$= 3$	3.3297	1.00	0.73	0.29	0.07	0.010	0.00	0.38	0.77	1.16	1.55
$\alpha = 5, \sigma = 3$	3.8433	1.00	0.77	0.35	0.09	0.016	0.00	0.72	5.45	8.18	10.90

Table-4: Bayes estimators of θ for p = 2, k = 1

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