Interval Estimation of the Stress-Strength Reliability in Lehmann Family of Distributions

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Abstract

This paper presents a unified approach for computing confidence limits for stress–strength reliability when strength and stress are independent random variables following a distribution in Lehmann family. The generalized confidence interval and the bootstrap confidence intervals are obtained. Simulation studies are conducted to assess the performance of the proposed methods in terms of the estimated coverage probabilities and the length of the confidence intervals. An example is also provided for illustration.

Keywords: Stress-strength reliability, Lehmann family, generalized pivotal quantity, boot-strap confidence interval.

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1 Introduction

Suppose the lifetime of a certain item is an absolutely continuous random variable with pdf $f(\cdot)$ and CDF $F(\cdot)$. Then the hazard and reversed hazard rate at the instant t are defined respectively as follows:

$$\mathrm{h(t)} = \frac{\mathrm{f(t)}}{1-\mathrm{F(t)}} \quad \text{and} \quad \mathrm{r(t)} = \frac{\mathrm{f(t)}}{\mathrm{F(t)}};\ \ \mathrm{t} > 0. \tag{1}$$

It may also be noted that the hazard function or the reversed hazard function uniquely determines the distribution function. From the literature it follows that a class of proportional hazard and reversed hazard models with CDF $G(t, \alpha)$ can be expressed as follows:

$$\mathrm{G(t,\alpha)} = [\mathrm{F(t)}]^{\alpha} \quad \text{and} \quad \mathrm{G(t,\alpha)} = 1 - [1-\mathrm{F(t)}]^{\alpha};\ \ \mathrm{t} > 0. \tag{2}$$

As stated by Balakrishnan (2021), the class of distributions defined in (2) can be written in a unified form as follows:

$$G^{*}(t,\alpha,\beta) = \{1-[1-F(t)]^{\alpha}\}^{\beta};\ \ \alpha,\beta > 0 \tag{3}$$

Note that (1) and (2) can be defined over the entire real line as well. As we consider only the life times were strict our attention to $t > 0$. $G(t,\alpha)$ has the hazard rate $\alpha h(t)$ and reversed hazard rate $\alpha r(t)$, a proportional hazard and reversed hazard rate with regard to that of $F(t)$. Therefore, α is termed as the proportionality parameter that is linked to the covariates in the data through a log-linear function. For more details of proportional hazard and reversed hazard families one may refer Kalbfleisch and Lawless (1991), Marshall and Olkin(2007) and Seo and Kim(2020).

The two classes of distributions defined in (2) were originally proposed by Lehman (1953) in testing of hypothesis to test $H_0 : F(x) = G(x)$ vs the alternative $H_1 : F(x) \leq G(x)$ for all x. He proposes to take $G(x) = [F(x)]^{\alpha}$. As α increases, the distribution G will shift more and more to the right of F. Any how α will capture the distance between the distributions $F(\cdot)$ and $G(\cdot)$. If α is an integer G is nothing but the distribution function of maximum, say V, of α iid random variables, say X, with distribution function F. Then $\mathrm{P(X < V)} = \frac{\alpha}{\alpha+1}$. If $\mathrm{G} = 1 - (1-F)^{\alpha}$ then the distribution of maxima of α iid random variables from F will switch to that of minima. The class of distributions given in (3) is commonly referred to as Lehmann families or Lehmann alternatives. $G(\cdot)$ is nothing but it is the distribution function of the exponentiated class of distributions with baseline distribution $F(\cdot)$. Though all exponentiated distributions belong to Lehmann family, each has its own intricacies in inferential aspects when used for modeling purpose. Note that the Lehmann families or Lehmann alternatives correspond to the proportional hazards and reversed hazards distributions.

Kundu and Gupta (2004) obtained some characterizations of the proportional hazard class and the proportional reversed hazard class of distributions and hence the Lehmann family also holds these properties. Let X and Y are two random variables with distribution functions F and G respectively and such that $\mathrm{G(t)} = [\mathrm{F(t)}]^{\alpha}$. Let $U(t) = -\ln F(t), a_Y^{(n)}(t) = E(U^n(Y)|Y < y)$ where $U^n(\cdot)$ denotes the nth power of $\mathrm{U}(\cdot)$. Let $\mathrm{r_X(t)}$ and, $\mathrm{r_Y(t)}$ be the reversed hazard rates with regard to X and Y respectively. Then there exist the following properties:

Property 1:

For any real number t such that $\mathrm{F(t)} > 0$, $\mathrm{r_Y(t)} = \alpha \mathrm{r_X(t)}$ with $\alpha > 0$ iff

$$a_Y^{(n)}(t) = U^n(t) + \frac{n}{\alpha} a_Y^{(n-1)}(t),$$

where n is a positive integer.

Property 2:

For any real number t such that $F(t) > 0, r_Y(t) = \alpha r_X(t)$ with $\alpha > 0$ iff

$$Var(U(Y)|Y < t) = \frac{1}{\alpha^2}.$$

The failure of a device occurs when the stress applied to it exceeds the strength. Let X and Y be two independent random strength and stress variables respectively. Then the function $R = P(Y < X)$ is called the stress-strength reliability which can be considered as a measure of performance of the device. Suppose X and Y have independent distributions belonging to Lehmann family which follow $G(x, \alpha_1)$ and $G(y, \alpha_2)$ respectively. Then R has the expression:

$$\mathrm{R} = \mathrm{P(Y < X)} = \iint_{\mathrm{y<x}} \mathrm{dG(x}, \alpha_1) \mathrm{dG(y}, \alpha_2) = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \qquad (4)$$

Different methods to estimate R can be seen in the literature, see for example, Church and Harris (1970), Cortese and Ventura (2013) and Jana et al. (2019). Interval estimation of R can be seen from Roy and Mathew (2003), Baklizi (2013), Wang et al. (2015), Xavier and Jose (2021), to mention a few. The present study deals with the interval estimation of R in Lehmann family using the generalized confidence interval proposed by Weerahandi (1993) and the bootstrap percentile methods.

2 Interval Estimation of *R* in Lehmann Family

Confidence intervals assure that a desired proportion of inferences about an uncertain parameter based on a random sample of given size will bound the true value so long as the distributional assumptions hold. In this study we consider the generalized confidence interval and the bootstrap percentile confidence interval as these two methods perform well among all other interval estimation methods in terms of coverage probabilities. The performance of the proposed methods are also assessed using simulation. Next, we shall provide a general procedure of aforementioned methods to avoid doing particular derivation and computation for each member in the Lehmann family. The two methods are illustrated in the case of four distributions in Lehmann family, namely, Topp-Leone, Burr III, Burr X and Power function distributions.

2.1 Generalized Confidence Interval

Let $(X_1, \ldots, X_n)$ be a random sample of size n from a distribution in the Lehmann family. Then the joint distribution function of $(X_1, \ldots, X_n)$ is the following:

$$\mathrm{G}(\mathbf{x}, \alpha) = [\mathrm{F}(\mathbf{x})]^{\alpha},$$

Where $\mathbf{x} = (x_1, \ldots, x_n)$ is the observed value of $(X_1, \ldots, X_n)$. Therefore

$$-2\sum_{i=1}^{n} \ln G(X_i, \alpha) \sim \chi^2(2n)$$

or

$$-2\alpha\sum_{i=1}^{n} \ln F(X_i) \sim \chi^2(2n).$$

Suppose two independent random samples of sizes n_i, say X_{ij}, for $j = 1, 2, \ldots, n_i$ and $i = 1, 2$ are taken from two independent populations in the Lehmann family with parameters $\alpha_i, i = 1, 2$ respectively. Define two statistics:

$$U_i = -2\alpha_i \sum_{j=1}^{n_i} \ln(F(X_{ij})) \sim \chi^2(2n_i),\ i = 1, 2.$$

As per the substitution method proposed by Weerahandi (2004), the generalized pivotal quantity (GPQ) for the parameters α_i for $i = 1, 2$ are the following:

$$T_{\alpha_i} = \frac{U_i}{-2\sum_{j=1}^{n_i} \ln(F(X_{ij}))}.$$

The GPQ of R, say T_R, obtained by substituting the respective GPQs of the parameters $\alpha_i, i = 1, 2$ in (4), is given below:

$$T_R = \frac{T_{\alpha_1}}{T_{\alpha_1} + T_{\alpha_2}}.$$

For a given data set of sample size (n_1, n_2) generate independently $U_i \sim \chi^2(2n_i), i = 1, 2$. Using these values compute $T_\alpha i$ and then compute T_R. This process of generating the value of T_R is repeated 10,000 times for the fixed values of (n_1, n_2). Based on the generated values of T_R, the percentiles of T_R can be estimated. If $T_R(\delta)$ is the $100\delta\%$ percentile of T_R then $(T_R(\delta/2), T_R(1-\delta/2))$ is the $100(1-\delta)\%$ confidence interval for R.

2.2 Bootstrap Confidence Interval

To construct bootstrap confidence interval, generate B bootstrap samples X_{ij}^* for $j = 1, \ldots, n_i; i = 1, 2$ from both populations and calculate the bootstrap estimates of the parameters, say $\hat{\alpha}_i^*$ for $i = 1, 2$. Using these bootstrap estimates of the parameters, compute the bootstrap estimate of R, say $\hat{R}^*$, using the expression:

$$\widehat{\mathrm{R}}^* = \frac{\widehat{\alpha}_1^*}{\widehat{\alpha}_1^* + \widehat{\alpha}_2^*}$$

For all bootstrap samples compute $\widehat{\mathrm{R}}^*$ for each sample. Then B(δ/2)th and B(1 − δ/2)th percentiles of $\widehat{\mathrm{R}}^*$ provides the 100(1 − δ)% bootstrap confidence limits for R. One can refer Efron and Tibishirani (1993) for more details on bootstrap methods.

Let us consider some special distributions in the Lehmann family of distributions. Estimators of the parameters of each distribution using GPQ and bootstrap percentile methods are given in the following table.

Table 1 GPQ and Bootstrap estimates of the parameters of some distributions in Lehmann family

		Estimates of $\alpha_i, i = 1, 2$	
Distribution	$G(x;\alpha); \alpha > 0$	GPQ(T_{α_i})	Bootstrap($\hat{\alpha}_i^*$)
Topp-Leone	$[x(2-x)]^{\alpha}; 0 < x < 1$	$\frac{U_i}{-2\sum_{j=1}^{n_i} \ln[x_{ij}(2-x_{ij})]}$	$\frac{n-(n_i-1)}{\sum_{j=1}^{n_i} \ln(x^*{}_{ij})+\sum_{j=1}^{n_i} \ln(2-x^*{}_{ij})}$
BurrIII	$(1+x^c)^{-\alpha}; x > 0$	$\frac{U_i}{2\sum_{j=1}^{n_i} \ln(1+x_{ij}{}^{-c})}$	$\frac{n_i-1}{\sum_{j=1}^{n_i} \ln(1+x^*{}_{ij}{}^{-c})}$
BurrX	$(1-e^{x^2})^{-\alpha}; x > 0$	$\frac{U_i}{-2\sum_{j=1}^{n_i} \ln(1-e^{-x^2}{}_{ij})}$	$\frac{-(n_i-1)}{\sum_{j=1}^{n_i} \ln(1-e^{-x^{*2}}{}_{ij})}$
PowerFunction	$x^{\alpha}; x > 0$	$\frac{U_i}{-2\sum_{j=1}^{n_i} \ln(x_{ij})}$	$\frac{n-(n_i-1)}{\sum_{i=1}^{n_i} \ln(x^*{}_{ij})}$

3 Simulation Study

A simulation study is conducted to assess the performance of the GPQ and percentile bootstrap methods to construct confidence interval for the stress-strength reliability of the distributions. The results are obtained using 10,000 simulated samples and computed using R codes. The generalized confidence interval for each simulated sample is computed using 10,000 values of the GPQ. For the bootstrap methods, 10,000 parametric bootstrap samples are used.

Table 2 Coverage probabilities and expected lengths of CIs of R in the case of Topp Leone distribution

		GPQ Method		Bootstrap Percentile Method	
Parameters	(n_1, n_2)	Coverage	Length	Coverage	Length
	(20,20)	0.9936	0.4078	0.955	0.2966
$\alpha_1 = 1$	(20,30)	0.9943	0.3767	0.933	0.2717
	(50,40)	0.9936	0.2841	0.955	0.2030
$\alpha_2 = 1$	(50,50)	0.9943	0.2688	0.96	0.1922
	(100,100)	0.9943	0.1927	0.955	0.1366
R = 0.5	(200,150)	0.9956	0.1481	0.965	0.1048
	(20,20)	0.9936	0.1708	0.949	0.1192
$\alpha_1 = 18$	(20,30)	0.9943	0.1565	0.945	0.1071
	(50,40)	0.992	0.1098	0.948	0.0761
$\alpha_2 = 2$	(50,50)	0.9916	0.1034	0.955	0.0720
	(100,100)	0.995	0.0718	0.947	0.0499
R = 0.9	(200,150)	0.9903	0.0545	0.945	0.0381
	(20,20)	0.9933	0.1705	0.962	0.1171
$\alpha_1 = 1$	(20,30)	0.9936	0.1549	0.944	0.1067
	(50,40)	0.996	0.1096	0.942	0.0767
$\alpha_2 = 9$	(50,50)	0.9906	0.1035	0.951	0.0720
	(100,100)	0.9953	0.0720	0.952	0.0504
R = 0.1	(200,150)	0.994	0.0539	0.943	0.0383

Table 3 Coverage probabilities and expected lengths of CIs of R in the case of BurIII distribution

		GPQ Method		Bootstrap Percentile Method	
Parameters	(n_1, n_2)	Coverage	Length	Coverage	Length
	(20,20)	0.952	0.2970	0.9456	0.2973
$\alpha_1 = 1$	(20,30)	0.9556	0.2732	0.9484	0.2731
	(50,40)	0.9527	0.2029	0.9492	0.2038
$\alpha_2 = 1$	(50,50)	0.951	0.1924	0.9522	0.1926
	(100,100)	0.9503	0.1372	0.9544	0.1373
R = 0.5	(200,150)	0.9493	0.1051	0.942	0.1052
	(20,20)	0.955	0.1190	0.9482	0.1183
$\alpha_1 = 18$	(20,30)	0.95	0.1078	0.9512	0.1072
	(50,40)	0.9483	0.0762	0.9522	0.0766
$\alpha_2 = 2$	(50,50)	0.9486	0.0722	0.9454	0.0721
	(100,100)	0.952	0.0503	0.9518	0.0505
R = 0.9	(200,150)	0.9486	0.0382	0.9526	0.0383
	(20,20)	0.9486	0.1184	0.949	0.1192
$\alpha_1 = 1$	(20,30)	0.945	0.1062	0.9544	0.1056
	(50,40)	0.9503	0.0768	0.9476	0.0771
$\alpha_2 = 9$	(50,50)	0.9496	0.0723	0.953	0.0722
	(100,100)	0.9486	0.0504	0.9502	0.0505
R = 0.1	(200,150)	0.9486	0.0383	0.9524	0.0383

Table 4 Coverage probabilities and expected Lengths of CIsof R in the case of Burr X distribution

		GPQ Method		Bootstrap Percentile Method	
Parameters	(n_1, n_2)	Coverage	Length	Coverage	Length
	(20,20)	0.947	0.2957	0.947	0.2962
$\alpha_1 = 1$	(20,30)	0.946	0.2720	0.948	0.2719
	(50,40)	0.941	0.2030	0.948	0.2029
$\alpha_2 = 1$	(50,50)	0.939	0.1916	0.945	0.1919
	(100,100)	0.948	0.1370	0.938	0.1366
R = 0.5	(200,150)	0.957	0.1048	0.949	0.1047
	(20,20)	0.952	0.1178	0.949	0.1175
$\alpha_1 = 18$	(20,30)	0.966	0.1058	0.944	0.1075
	(50,40)	0.954	0.0762	0.949	0.0761
$\alpha_2 = 2$	(50,50)	0.939	0.0714	0.957	0.0719
	(100,100)	0.943	0.0500	0.944	0.0506
R = 0.9	(200,150)	0.958	0.0380	0.945	0.0384
	(20,20)	0.948	0.1179	0.948	0.1172
$\alpha_1 = 1$	(20,30)	0.952	0.1057	0.954	0.1060
	(50,40)	0.952	0.0761	0.949	0.0760
$\alpha_2 = 9$	(50,50)	0.955	0.0722	0.944	0.0710
	(100,100)	0.942	0.0502	0.948	0.0501
R = 0.1	(200,150)	0.931	0.0383	0.946	0.0382

Table 5 Coverage probabilities and expected lengths CIs of R in the case of Power function distribution

		GPQ Method		Bootstrap Percentile Method	
Parameters	(n_1, n_2)	Coverage	Length	Coverage	Length
	(20,20)	0.9773	0.3479	0.935	0.2958
$\alpha_1 = 1$	(20,30)	0.9803	0.3203	0.952	0.2724
	(50,40)	0.9806	0.2403	0.947	0.2031
$\alpha_2 = 1$	(50,50)	0.9726	0.2271	0.953	0.1920
	(100,100)	0.9803	0.1624	0.948	0.1368
R = 0.5	(200,150)	0.978	0.1245	0.933	0.1047
	(20,20)	0.9806	0.1419	0.929	0.1174
$\alpha_1 = 18$	(20,30)	0.9846	0.1295	0.952	0.1087
	(50,40)	0.9766	0.0919	0.956	0.0765
$\alpha_2 = 2$	(50,50)	0.9763	0.0862	0.948	0.0717
	(100,100)	0.9803	0.0600	0.948	0.0503
R = 0.9	(200,150)	0.9796	0.0456	0.944	0.0381
	(20,20)	0.9816	0.1426	0.946	0.1171
$\alpha_1 = 1$	(20,30)	0.978	0.1275	0.937	0.1057
	(50,40)	0.9736	0.0917	0.953	0.0761
$\alpha_2 = 9$	(50,50)	0.979	0.0863	0.949	0.0716
	(100,100)	0.983	0.0599	0.943	0.0502
R = 0.1	(200,150)	0.9766	0.0456	0.946	0.0382

4 An Example

To illustrate the proposed methods of interval estimation of R, a real data reproduced from Condino et al. (2016). The values are the time taken to score the first goal during the final stage of soccer matches of the European Champions league for two consecutive years (2011–12 and 2012-13). All times are divided by 90, that is, the total time of a soccer match in minutes so that all values belong to (0,1).

Data Set I: First Matches

0.033	0.111	0.344	0.222	0.078	0.622	0.133	0.633	0.422	0.011	0.100	0.278
0.089	0.500	0.822	0.833	0.489	0.644	0.456	0.222	0.167	0.311	0.300	0.956

Data Set I: Return Matches

0.267	0.611	0.344	0.533	0.033	0.478	0.200	0.056	0.556	0.711	0.078	0.533
0.922	0.067	0.389	0.289	0.233	0.144	0.100	0.278	0.500	0.078	0.289	0.311
0.122											

Topp-Leone is fitted for both data sets by treating data set I and data set II as realizations of independent random samples from X (strength) and Y (stress) respectively. The estimated values of the parameters are $\widehat{\boldsymbol{\alpha}}_{\mathbf{1}} = 1.0740$

and $\widehat{\boldsymbol{\alpha}}_{\mathbf{2}} = 1.0713$ respectively. The distributions fit well for the datasets at 5% level of significance. The estimated value of R is 0.5006. The confidence intervals based on GPQ and bootstrap methods are (0.3163, 0.6941) and (0.3902, 0.6643) respectively. It may be noted that the bootstrap method provides shortest length confidence interval.

5 Conclusion

Simulation study shows that bootstrap confidence interval provides better coverage confidence intervals for all the parameter combinations and all sample sizes in the cases of Topp-Leone and power function distributions. In these cases, the expected lengths provided by these two methods are not comparable because of the unsatisfactory coverage probabilities given by GPQ method. For the Burr III and Burr X distributions, both the methods provide satisfactory coverages for all the parameter combinations. Both the methods give equal length confidence intervals also. So, our overall recommendation is that percentile bootstrap method is preferable to the GPQ method. As $|\alpha_1 - \alpha_2|$ increases the distance between the CDFs of stress and strength also will increase. Naturally, the expected length of the confidence intervals will decreaseas $|\alpha_1 - \alpha_2|$ increases.

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