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# The Predicted Failure on A Two-Dimensional Warranty Using the Bayesian Approach

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## **Abstract**

Traditionally, the warranty cost is assumed to be the cost of repairs based on the average cost that arises from a damage claim. Several studies have considered the Least Square and Maximum Likelihood Estimation methods in estimating the parameters of the failure distribution. However, this study uses the Bayesian method using the posterior distribution obtained from the prior distribution and the likelihood function. The Bayesian approach is more optimal to use in estimating parameters because it has the smallest value of Aikake's Information Criterion (AIC) compared to other methods. Failure expectations that are close to natural can be used to analyze survival to determine how long a product will last before the failure. The numerical example in this study, is the type of motorcycle with an engine capacity of

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125 CC with the Weibull distribution, while the 150 CC and 160 CC with Exponential distribution. The novelty in this study is that the free repair approach in two-dimensional can be anticipated with failure considering the dimensions of age and mileage.

**Keywords:** Warranty, failure, Bayesian, two-dimensional.

## 1 Introduction

Transportation has a key role in the success of development, especially when it comes to promoting local economic activity. The motorbike is one of the most popular ways of transportation. Motorbike were chosen because they are more widely accessible, affordable, and useful for everyday use. According to the data from Badan Pusat Statistik (BPS), the Central Bureau of Statistics in Indonesia, motorbike usage climbed from 126,508,776 in 2018 to 136,137,451 in 2020 [1]. As a result, businesses competed with one another to boost product sales. To combat this, several businesses offer warranty services for goods that are harmed within a specific time frame. A warranty is a promise made by the business to compensate you for damaged goods if it occurs within the time frame specified in the contract [2].

The warranty for motorbikes has a two-dimensional warranty structure, where the warranty structure is determined by the motorbike's age and mileage. The consideration of a product's ability to survive until failure is strongly tied to the offering of guarantees. The survival of a population or sample that is observed may be determined by estimating the parameters of the distribution if the specific distribution of the data is known. Parameter estimation can be done in a variety of ways. In 2008, Fang Huang [3] used the Bayesian approach to estimate parameters in his research to find the best pricing and production schedule for a corporation. Subsequent research was conducted by Turkson [4] in 2022 to compare Maximum Likelihood Estimation (MLE) and Bayesian method for parameter estimation on a product's ability to survive until failure in determining one-dimensional policy warranty. Weibull and Lognormal distributions were chosen for this study.

These two types of warranties provide vehicle/motorbike with different forms of protection, and their coverage may overlap or complement each other depending on the manufacturer's warranty policy. Mileage warranties are particularly important for high-mileage vehicles, while age warranties

ensure that even low-mileage vehicles are covered for a certain period, considering the natural aging of components. There are two types of vehicle warranties: one based on mileage, and another based on the age of the vehicle. In the context of motorbikes, there are two common types of warranties that are often associated with mileage and age: For example, a common vehicle warranty might cover the engine and transmission for 5 years or 60,000 miles, whichever comes first. This means that if we reach the mileage limit before the time limit, the warranty will expire. There are two types of warranties: express warranties, which are specific promises made by sellers, and implied warranties, which are automatically imposed by law to protect consumers.

When a product is sold with a warranty, a separate charge known as the warranty fee must be paid. Various models are employed to determine the warranty cost in line with each company's rules. In 2004, Baik, et al. [5] carried out research to estimate the product's lifetime and warranty costs under a two-dimensional warranty policy with few repairs. The look into the matter was then carried out by Bai and Pham [6] in 2005 using repair limit risk-free warranty concept or repair risk-free limit with the assumption of imperfect repair to obtain a one-dimensional policy warranty cost estimate. In addition, Darghouth [7] carried out research in 2017 to develop a warranty cost model that included a preventative maintenance policy in a one-dimensional warranty policy. Lukitosari, et al. [8, 9] also discussed strategies to support uncertainty inventory due to damage. But no one has included failure predictions in determining the warranty price.

The data to be used in this paper include information on the age and mileage of motorbikes with 125 CC, 150 CC, and 160 CC machine capacity. A variety of methods, including the Least Square approach, the Maximum Likelihood Estimation (MLE) method, and the Bayesian method, are used to calculate parameter estimates for the chosen distribution. The parameter result that has the smallest error will be chosen among the three methods' outcomes. A preventative maintenance program in the form of routine servicing will be used to calculate the addition of warranty costs.

## **2 Preliminary Study**

### **2.1 Survival Analysis**

In survival analysis, the term "survival time" typically refers to the length of time an individual survived during the observation period [10].

### 1. Probability Density Function

The Probability Density Function (PDF) is a function contained in a continuous random variable. The format of PDF is as follows [11]:

$$f(x) = \frac{d}{dx}F(x) \quad (1)$$

with cumulative distribution function:

$$F(x) = \int_{-\infty}^x f(t) dt, \text{ for } -\infty < x < \infty \quad (2)$$

For probability density function of the Weibull distribution can be defined as:

$$f(x) = \begin{cases} \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}, & 0 < x < \infty \\ 0, & \text{other } x \end{cases} \quad (3)$$

As for the probability density function of the Exponential distribution:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \text{ other} \end{cases} \quad (4)$$

### 2. Survival Function

The survival function denoted by  $S(x)$  defines the probability of an individual surviving more than the specified time  $x$ . The survival function equation can be written as follows [10]:

$$\begin{aligned} S(x) &= P(X > x) \\ &= 1 - P(X \leq x) \\ &= 1 - F(x) \end{aligned} \quad (5)$$

For survival function of Weibull distribution can be defined as:

### 3. Hazard Function (Failure Rate)

Hazard Function or denoted with  $h(x)$  which is defined as the rate of an individual experiencing failure between time intervals. Hazard function can be written as [10]:

$$h(x) = \frac{f(x)}{S(x)} \quad (6)$$

## 2.2 Warranty

A warranty is a promise made by a company to replace a product if it is damaged within a specified time frame as specified in particular agreements [2]. The company will replace or repair any products that are damaged during the warranty period at no cost to the customer. The warranty policies are divided into the following categories based on how many factors affect the length of the guarantee [12]:

1. One-dimensional warranty policy  
This policy applies when a corporation limits the warranty period based solely on one factor (e.g., age, frequency of use, results of use, etc.).
2. Two-dimensional warranty policy  
This policy applies when a corporation limits the warranty period based on two variables. For instance, the organization employs the time and mileage factors found in motorbike warranties.

A motorbike warranty has a two-dimensional warranty policy, so the first failure's age and use are indicated by the numbers  $x$  and  $y$ . Based on Equation (1), the two-dimensional guarantee policy's opportunity density function is a combination function with the variable  $(x, y)$  that is written as follows [11]:

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} \quad (7)$$

with the cumulative distribution function based on Equation (2) as follows:

$$F(x, y) = P(X \leq x, Y \leq y); x \geq 0, y \geq 0.$$

Based on Equation (5), the survival function of the variables  $x$  and  $y$  is formulated as follows:

$$S(x, y) = 1 - F(x, y) \quad (8)$$

Whereas for hazard function is based on Equation (6):

$$h(x, y) = \frac{f(x, y)}{S(x, y)} \quad (9)$$

And as for the expectation of failure quantity on a two-dimensional approach from  $[0, x) \times [0, y)$  can be written as follows [13]:

$$M(u, w) = \int_0^w \int_0^u h(x, y) dx dy \quad (10)$$

### 2.3 Least Square Method

The Least Square Method is one of the methods used to estimate or estimate the failure distribution parameters. The Least Square method is used to estimate the linear regression coefficient [14].

Using the following equation, the Weibull distribution's parameter estimation can be obtained [14]:

$$\begin{aligned}\hat{\beta} &= b \\ \hat{\alpha} &= e^{-\frac{a}{b}}\end{aligned}$$

with,

$$\begin{aligned}a &= \frac{1}{n} \left( \sum_{i=1}^n y_i - b \sum_{i=1}^n x_i \right) \\ &= \bar{Y} - b\bar{X} \\ b &= \frac{\sum_{i=1}^n x_i y_i - \bar{Y} \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2 - \bar{X} \sum_{i=1}^n x_i}\end{aligned}$$

In contrast, the exponential distribution is as follows [14]:

$$\hat{\lambda} = b$$

with,

$$b = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

### 2.4 Maximum Likelihood Estimation Method (MLE)

To obtain parameter estimation using the MLE method, it is done by establishing a likelihood function based on the data distribution and looking for the ln likelihood value. Then the ln likelihood function is differentiated against the  $f(x_i, y_i)$  function and equated to zero. The likelihood function can be formulated with [15]:

$$L(\theta; t) = \prod_{i=1}^n f(t_i) \quad (11)$$

Using the MLE approach, parameter estimation is as follows:

1. Weibull Distribution

Based on Equation (3), the probability density function is obtained for the Weibull distribution with the variables  $x$  and  $y$  is as follows:

$$f(x, y) = \frac{\beta_1 \beta_2}{\alpha_1^{\beta_1} \alpha_2^{\beta_2}} e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1} - \left(\frac{y}{\alpha_2}\right)^{\beta_2}} x^{\beta_1-1} y^{\beta_2-1} \quad (12)$$

The likelihood function is obtained according to Equation (11) as follows:

$$\begin{aligned} L(\hat{\theta}) &= \prod_{i=1}^n f_i(x, y) \\ &= \left( \frac{\beta_1 \beta_2}{\alpha_1^{\beta_1} \alpha_2^{\beta_2}} \right)^n e^{\sum_{i=1}^n -\left(\frac{x_i}{\alpha_1}\right)^{\beta_1} - \left(\frac{y_i}{\alpha_2}\right)^{\beta_2}} \prod_{i=1}^n x_i^{\beta_1-1} y_i^{\beta_2-1} \end{aligned}$$

with  $\hat{\theta} = \alpha_1, \alpha_2, \beta_1, \beta_2$

Then the ln likelihood function is obtained:

$$\begin{aligned} \ln L(\hat{\theta}) &= n(\ln \beta_1 + \ln \beta_2 - \beta_1 \ln \alpha_1 - \beta_2 \ln \alpha_2) + (\beta_1 - 1) \sum_{i=1}^n \ln(x_i) \\ &\quad + (\beta_2 - 1) \sum_{i=1}^n \ln(y_i) - \sum_{i=1}^n \left(\frac{x_i}{\alpha_1}\right)^{\beta_1} - \sum_{i=1}^n \left(\frac{y_i}{\alpha_2}\right)^{\beta_2} \end{aligned}$$

To get an estimation of  $\alpha_1, \alpha_2, \beta_1,$  and  $\beta_2$  parameters,  $\ln$  likelihood function is derived from these parameters and then equalized to zero. The outcomes of the non-linear equations are determined for the Weibull distribution scenario, and Matlab is used to calculate the parameters using the Newton-Raphson method.

2. Exponential Distribution

Based on Equation (4), the probability of density function for the Exponential distribution with  $x$  and  $y$  variables is:

$$f(x, y) = \lambda_1 \lambda_2 e^{-\lambda_1 x_i - \lambda_2 y_i} \quad (13)$$

Obtained likelihood function fit to Equation (11):

$$\begin{aligned} L(\lambda_1, \lambda_2) &= \prod_{i=1}^n f_i(x, y) \\ &= (\lambda_1 \lambda_2)^n e^{\sum_{i=1}^n -\lambda_1 x_i - \lambda_2 y_i} \end{aligned} \quad (14)$$

Then ln likelihood function is obtained:

$$\ln L(\lambda_1 \lambda_2) = n \ln(\lambda_1 \lambda_2) - \lambda_1 \sum_{i=1}^n x_i - \lambda_2 \sum_{i=1}^n y_i$$

To get the estimation of  $\lambda_1 \lambda_2$  parameter, ln likelihood function is derived from these parameters and then equalized to zero. The following is the derived equation:

$$\begin{aligned} \frac{\partial \ln L(\lambda_1 \lambda_2)}{\partial \lambda_1} &= 0 \\ \hat{\lambda}_1 &= \frac{n}{\sum_{i=1}^n x_i} \\ \frac{\partial \ln L(\lambda_1 \lambda_2)}{\partial \lambda_2} &= 0 \\ \hat{\lambda}_2 &= \frac{n}{\sum_{i=1}^n y_i} \end{aligned}$$

## 2.5 Bayesian Method

The Bayesian technique uses a conditional probability definition and follows the Bayes rule [16]. Before computing the data, the Bayesian technique performs a prior distribution calculation as basic knowledge. The posterior distribution is then created by combining the prior distribution and likelihood function. The Jeffrey's distribution must be utilized as the beginning distribution since the non-informative previous distribution is employed in this investigation [17].

### 1. Weibull Distribution

According to the probability density function on Equation (12), the probability density function can be obtained by transforming the function to Exponential form by  $\omega_1 = \frac{1}{\alpha_1^{\beta_1}}$  and  $\omega_2 = \frac{1}{\alpha_2^{\beta_2}}$ , then:

$$f(x, y) = \beta_1 \omega_1 x^{\beta_1 - 1} e^{-\omega_1 x^{\beta_1}} \beta_2 \omega_2 y^{\beta_2 - 1} e^{-\omega_2 y^{\beta_2}}$$

By doing the separation of  $r = x^{\beta_1}$  and  $s = y^{\beta_2}$ , then the transformation is:

$$f(r, s) = \omega_1 e^{-\omega_1 r} \omega_2 e^{-\omega_2 s}$$

Next, the ln  $f(r, s)$  function is obtained:

$$\ln f(r, s) = \ln(\omega_1 \omega_2) - \omega_1 r - \omega_2 s$$



Then, differentiating  $\ln f(r, s)$  to  $\omega_1$  and  $\omega_2$ :

$$\begin{aligned} \frac{\partial^2 \ln f(r, s)}{\partial \omega_1^2} &= -\frac{1}{\omega_1^2} & \frac{\partial^2 \ln f(r, s)}{\partial \omega_2^2} &= -\frac{1}{\omega_2^2} \\ \frac{\partial^2 \ln f(r, s)}{\partial \omega_2 \omega_1} &= 0 & \frac{\partial^2 \ln f(r, s)}{\partial \omega_1 \omega_2} &= 0 \end{aligned}$$

Next, determine each matrix element's anticipated value using:

$$\begin{aligned} E\left(-\frac{1}{\omega_1^2}\right) &= -\frac{1}{\omega_1^2} \\ E\left(-\frac{1}{\omega_2^2}\right) &= -\frac{1}{\omega_2^2} \end{aligned}$$

as a result, the following Fisher information was obtained::

$$I(\omega_1 \omega_2) = -n \begin{bmatrix} -\frac{1}{\omega_1^2} & 0 \\ 0 & -\frac{1}{\omega_2^2} \end{bmatrix} = \begin{bmatrix} \frac{n}{\omega_1^2} & 0 \\ 0 & \frac{n}{\omega_2^2} \end{bmatrix}$$

Prior distribution can be obtained by using Jeffrey's distribution rule:

$$p(\omega_1, \omega_2) = \sqrt{|I(\omega_1 \omega_2)|} = \frac{n}{\omega_1 \omega_2}$$

The following is how to obtain the likelihood function:

$$L(\omega_1, \omega_2) = (\omega_1 \omega_2)^n e^{-\omega_1 (\sum_{i=1}^n r_i) - \omega_2 (\sum_{i=1}^n s_i)}$$

The posterior distribution will then be calculated as follows:

$$\begin{aligned} \pi(\omega_1 \omega_2) &= \frac{L(\omega_1 \omega_2) p(\omega_1 \omega_2)}{\int \int L(\omega_1 \omega_2) p(\omega_1 \omega_2) d\omega_1 d\omega_2} \\ &= \frac{(\sum_{i=1}^n x^{\beta_1})^n [\omega_1^{n-1} e^{\omega_1 \sum_{i=1}^n x^{\beta_1}}] (\sum_{i=1}^n y^{\beta_2})^n [\omega_2^{n-1} e^{\omega_2 \sum_{i=1}^n y^{\beta_2}}]}{\Gamma(n) \Gamma(n)} \end{aligned}$$

Thus, obtained parameter estimates for  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$ :

$$\begin{aligned} \hat{\alpha}_1 &= \left( \frac{\sum_{i=1}^n x^{\beta_1}}{n} \right)^{\frac{1}{\beta_1}} \\ \hat{\alpha}_2 &= \left( \frac{\sum_{i=1}^n y^{\beta_2}}{n} \right)^{\frac{1}{\beta_2}} \end{aligned}$$

with a reliance interval of  $100(1 - \alpha)\%$  for  $\omega_1$  and  $\omega_2$  as:

$$\left[ \frac{X_{\frac{\alpha}{2}}^2(2n)}{2 \sum x^{\beta_1}}, \frac{X_{1-\frac{\alpha}{2}}^2(2n)}{2 \sum x^{\beta_1}} \right]$$

## 2. Exponential Distribution

The following step is to determine the value of  $\ln f(x, y)$  as illustrated, using Equation (13)'s probability density and (14)'s likelihood functions, respectively:

$$\ln f(x, y) = \ln(\lambda_1 \lambda_2) - \lambda_1 x_i - \lambda_2 y_i$$

Differentiating  $\ln f(x, y)$  for  $\lambda_1$  dan  $\lambda_2$ :

$$\begin{aligned} \frac{\partial^2 \ln f(x, y)}{\partial \lambda_1^2} &= -\frac{1}{\lambda_1^2} & \frac{\partial^2 \ln f(x, y)}{\partial \lambda_2^2} &= -\frac{1}{\lambda_2^2} \\ \frac{\partial^2 \ln f(x, y)}{\partial \lambda_2 \lambda_1} &= 0 & \frac{\partial^2 \ln f(x, y)}{\partial \lambda_1 \lambda_2} &= 0 \end{aligned}$$

The expected value of each matrix member is then determined as follows:

$$\begin{aligned} E\left(-\frac{1}{\lambda_1^2}\right) &= -\frac{1}{\lambda_1^2} \\ E\left(-\frac{1}{\lambda_2^2}\right) &= -\frac{1}{\lambda_2^2} \end{aligned}$$

As a result, the following Fisher information was obtained::

$$I(\lambda_1 \lambda_2) = -n \begin{bmatrix} -\frac{1}{\lambda_1^2} & 0 \\ 0 & -\frac{1}{\lambda_2^2} \end{bmatrix} = \begin{bmatrix} n \frac{1}{\lambda_1^2} & 0 \\ 0 & n \frac{1}{\lambda_2^2} \end{bmatrix}$$

Prior distribution can be obtained by using Jeffrey's distribution rule:

$$p(\lambda_1, \lambda_2) = \sqrt{|I(\lambda_1 \lambda_2)|} = \frac{n}{\lambda_1 \lambda_2}$$

The posterior distribution will then be calculated as follows:

$$\begin{aligned} \pi(\lambda_1 \lambda_2) &= \frac{L(\lambda_1 \lambda_2) p(\lambda_1 \lambda_2)}{\int \int L(\lambda_1 \lambda_2) p(\lambda_1 \lambda_2) d\lambda_1 d\lambda_2} \\ &= \left( \frac{\lambda_1^{n-1} e^{-\lambda_1 u} u^n}{\Gamma(n)} \right) \left( \frac{\lambda_2^{n-1} e^{-\lambda_2 w} w^n}{\Gamma(n)} \right) \end{aligned}$$

Thus, obtained parameter estimates for  $\hat{\lambda}_1$  and  $\hat{\lambda}_2$ :

$$\hat{\lambda}_1 = \frac{n}{\sum_{i=1}^n x_i}$$

$$\hat{\lambda}_2 = \frac{n}{\sum_{i=1}^n y_i}$$

### 2.6 Warranty Cost

In this study, manufacturers implement a free repair/replacement warranty policy, under which any defective items will be fixed or replaced free of charge if they are discovered during the guarantee term. In order to lessen the chance of early failure, manufacturers also provide routine preventative maintenance services for a device during the warranty duration [7]. Therefore, the expected warranty costs can be calculated using the following equation [12]:

$$E[C_s(W)] = C_s M(W) + C_r \tag{15}$$

where,

- $E[C_s(W)]$  : Expected warranty cost
- $C_s$  : Optimum cost
- $C_r$  : Replacement cost
- $M(W)$  : Amount of failure expectation

Utilizing the following calculation, determine what the ideal cost would be:

$$C_s = \sum_{n=1}^n C_p \tag{16}$$

where,

- $C_p$  : Preventive maintenance costs

## 3 Results and Discussion

### 3.1 Data Description

The age and mileage information used in this study came from a service centre between January 2020 and December 2022. The type of motorbike used was an engine capacity of 125 CC, 150 CC and 160 CC.

### 3.2 Determination of Data Distribution

Data distribution is determined in two steps, first by computing the value of the Index of Fit and then by evaluating the distribution's appropriateness or Goodness of Fit.

#### 1. Index of Fit

By using the ( $r$ ) correlation coefficient equation [14]:

$$r = \frac{n \sum_{i=1}^n x_i y_i - (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{\sqrt{[n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2][n \sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2]}} \quad (17)$$

Initial identification of  $(x_i, y_i)$  value is carried out with a linear regression plot approach that approaches the data.  $(x_i, y_i)$  value for each distribution, as follows:

- Weibull Distribution

$$(x_i, y_i) = \left( \ln(t_i), \ln \ln \left[ \frac{1}{1 - F(t_i)} \right] \right)$$

- Exponential Distribution

$$(x_i, y_i) = \left( (t_i), \ln \left[ \frac{1}{1 - F(t_i)} \right] \right)$$

- Normal Distribution

$$(x_i, y_i) = (t_i, \phi^{-1}[F(t_i)])$$

- Lognormal Distribution

$$(x_i, y_i) = (\ln(t_i), \phi^{-1}[F(t_i)])$$

By using the value of  $F(t_i) = \frac{i-0,3}{n+0,4}$  and Equation (17), it is obtained the initial Index of Fit for each motorbike as seen in Table 1. The analysis has been conducted to determine the most suitable probability distribution for modeling the lifetime data of different motorcycle engine types (125 CC, 150 CC, and 160 CC). The choice of distribution is based on the Index of Fit ( $r$ ) values, where the distribution with the highest  $r$  value is considered the most suitable.

**Table 1** Index of Fit for Each Distribution

Capacity	Dimension	Distribution				largest r value
		Weibull	Exponential	Normal	Log normal	
125 CC	Age	0.9525	0.9114	0.9469	0.9482	Weibull
	Mileage	0.9881	0.9584	0.9224	0.9827	Weibull
150 CC	Age	0.8618	0.8861	0.8749	0.8791	Exponential
	Mileage	0.9022	0.9132	0.9131	0.9049	Exponential
160 CC	Age	0.8785	0.8973	0.8577	0.8806	Exponential
	Mileage	0.9123	0.9426	0.7786	0.9331	Exponential

The choice of distribution is based on how well the data fits the distribution’s assumptions. The distribution with the highest r value is considered to be the best fit for the data. The Weibull distribution is often used to model the lifetimes of products or components that have a wear-out phase, where the failure rate increases over time. This could be the case for the 125 CC engine, which may have a higher likelihood of failure as it ages. On the other hand, the Exponential distribution is commonly used when modeling lifetimes without a wear-out phase, where failures occur randomly at a constant rate. This may be a better fit for the 150 CC and 160 CC engines, suggesting that the failures are not necessarily related to aging but occur at a relatively constant rate. The choice of distribution should align with the characteristics of the data and the underlying mechanisms of failure. The higher r values suggest that the chosen distributions provide a better fit to the data for these engine types.

2. Goodness of Fit

Based on the results of the selected distribution obtained from the Index of Fit computation, the motorbike with 125 CC with Weibull distributed can be subjected to the Mann test while the motorbike with 150 CC and 160 CC with exponentially distributed can be subjected to Barlett test [14].

Hypothesis:

$H_0 : T = F(t)$  for Weibull distributed data

$H_1 : T \neq F(t)$  for non-Weibull distributed data

Statistic Test:

$$M = \frac{k_1 \sum_{i=k_1+1}^{n-1} \left[ \frac{\ln t_{t+1} - \ln t_i}{M_i} \right]}{k_2 \sum_{i=1}^{k_1} \left[ \frac{\ln t_{t+1} - \ln t_i}{M_i} \right]} \tag{18}$$

Test Criteria:

If  $M < F_{\alpha, v_1, v_2}$  then  $H_0$  is accepted and  $H_1$  is rejected with  $v_1 = 2 \left[ \frac{n}{2} \right]$ ;  $v_2 = 2 \left[ \frac{n-1}{2} \right]$

Table 2 shows the results of the Mann test calculation for a 125 CC motorbike using Equation (18).

**Table 2** Result of Mann Test Calculation

Capacity	Dimension	$v_1$	$v_2$	$F_{0.05; v_1; v_2}$	$M$
125 CC	Age	980	979	1.1109	0.6236
	Mileage	980	979	1.1109	1.0216

The following hypothesis was tested using the Barlett test [14]:

Hypothesis:

$H_0 : T = F(t)$  for exponentially distributed data

$H_1 : T \neq F(t)$  for non-exponentially distributed data

Statistic Test:

$$B = \frac{2n \left[ \ln \left( \frac{1}{n} \right) \sum_{i=1}^n t_i - \left( \frac{1}{n} \right) \sum_{i=1}^n t_i \right]}{1 + \frac{n+1}{6n}} \tag{19}$$

Test Criteria:

If  $X_{1-\frac{\alpha}{2}, n-1}^2 < B < X_{\frac{\alpha}{2}, n-1}^2$  the  $H_0$  is accept and  $H_1$  is rejected.

Table 3 shows the results of the Barlett test calculation for a 150 CC and 160 CC motorbike using Equation (19).

**Table 3** Result of Barlett Test

Capacity	Dimension	$n$	$X_{0.975, n-1}^2$	$B$	$X_{0.025, n-1}^2$
150 CC	Age	338	288.0363	296.5614	389.7507
	Mileage	338	288.0363	367.9689	389.7507
160 CC	Age	226	185.3483	199.04	268.4378
	Mileage	226	185.3483	192.9272	268.4378

### 3.3 Parameter Estimation

Parameter estimation is carried out using the Least Squares method, the Maximum Likelihood estimation method, and the Bayesian approach. Motorcycles with an engine size of 125 CC are Weibull distributed, but motorcycles with an machine capacity of 150 CC and 160 CC are distributed exponentially, as was discovered in the previous sub-chapter. Therefore, the parameter estimations shown in Table 4.

**Table 4** Parameter estimation of three methods

Capacity	Parameter	Method		
		Least Square	MLE	Bayesian
125 CC	$\beta_1$	1.550385	1.559355	1.559355
	$\beta_2$	1.239813	1.167246	1.167242
	$\alpha_1$	178.352123	179.132416	179.132405
	$\alpha_2$	3,950.29914	4,020.425086	4,020.425463
150 CC	$\lambda_1$	0.006992	0.007282	0.007282
	$\lambda_2$	0.000227	0.000234	0.000234
160 CC	$\lambda_1$	0.007485	0.008034	0.008034
	$\lambda_2$	0.000705	0.000816	0.000816

The use of the least squares method in regression analysis, particularly when dealing with lifetime or survival data, may not always yield the best results, especially when the underlying relationships are not linear. However, there are situations where researchers or practitioners choose to use the least squares method, even when other techniques like Maximum Likelihood Estimation (MLE) or Bayesian methods might be more appropriate.

Next, the calculation of Aikaike’s Information Criterion (AIC) for each parameter estimation method is performed with the following formula [11]:

$$AIC = -2l + 2p \tag{20}$$

where  $l$  is ln likelihood function and  $p$  is the number of parameters. Table 5 shows the results of the AIC calculation based on Equation (20).

**Table 5** AIC calculation of three methods

Capacity	AIC		
	Least Square	MLE	Bayesian
125 CC	29,767.8290599285	29,759.1756400028	29,759.1756400027
150 CC	10,336.9662904316	10,336.1413401608	10,336.1413401608
160 CC	6,308.3828113507	6,302.6702236124	6,302.6702236124

The Bayesian approach is shown to be more advantageous to utilize in parameter estimation based on the comparison of AIC values in Table 5, since it has the least value when compared to other approaches.

### 3.4 Expected of Failure

According to the distribution that is used, the hazard function or rate of failure is integrated to determine the predicted quantity of failure. The predicted

amount of failure is calculated using Equation (10) as follows:

$$M(u, w) = \int_0^w \int_0^u h(x, y) dx dy$$

The expected expression of the Weibull distribution of failure is achieved by substituting Equation (9) into Equation (10), such as:

$$\begin{aligned} M(u, w) &= \int_0^w \int_0^u \frac{\frac{\beta_1 \beta_2}{\alpha_1^{\beta_1} \alpha_2^{\beta_2}} x^{\beta_1-1} y^{\beta_2-1}}{e^{\left(\frac{y}{\alpha_2}\right)^{\beta_2}} + e^{\left(\frac{x}{\alpha_1}\right)^{\beta_1}} - 1} dx dy \\ &= \left(\frac{u}{\alpha_1}\right)^{\beta_1} \left( \ln \left| e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}} - 1 \right| - \left(\frac{w}{\alpha_2}\right)^{\beta_2} \right) \\ &\quad - \operatorname{dilog} \left( \frac{1 - e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}}}{e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}}} \right) \\ &\quad - \ln \left| e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}} + e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}} - 1 \right| \ln \left| \frac{1 - e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}}}{e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}}} \right| \\ &\quad + \operatorname{dilog} \left( \frac{e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}}}{1 - e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}}} \right) \\ &\quad + \ln \left| e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}} + e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}} - 1 \right| \ln \left| \frac{e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}}}{1 - e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}}} \right| \\ &\quad - \operatorname{dilog} \left( e^{\left(\frac{w}{\alpha_2}\right)^{\beta_2}} \right) \\ &\quad + \frac{1}{6} \pi^2 - \frac{1}{2} \left( \left(\frac{w}{\alpha_2}\right)^{\beta_2} \right)^2 - \operatorname{dilog} \left( \frac{1}{1 - e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}}} \right) \\ &\quad - \ln \left| e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}} \right| \ln \left| \frac{1}{1 - e^{\left(\frac{u}{\alpha_1}\right)^{\beta_1}}} \right| \end{aligned} \quad (21)$$

Whereas for the formulation of expectations for the amount of failure in the Exponential distribution, that is by substituting Equation (9) into



Equation (10) as follows:

$$\begin{aligned}
 M(u, w) &= \int_0^w \int_0^u \frac{\lambda_1 \lambda_2}{e^{\lambda_1 x} + e^{\lambda_2 y}} dx dy \\
 &= \lambda_1 u [\ln |e^{\lambda_2 w} - 1| - \lambda_2 w] - \operatorname{dilog} \left( \frac{1 - e^{\lambda_2 w}}{e^{\lambda_1 u}} \right) \\
 &\quad - \ln |e^{\lambda_1 u} + e^{\lambda_2 w} - 1| \ln \left| \frac{1 - e^{\lambda_2 w}}{e^{\lambda_1 u}} \right| + \operatorname{dilog} \left( \frac{e^{\lambda_2 w}}{1 - e^{\lambda_1 u}} \right) \\
 &\quad + \ln |e^{\lambda_1 u} + e^{\lambda_2 w} - 1| \ln \left| \frac{e^{\lambda_2 w}}{1 - e^{\lambda_1 u}} \right| - \operatorname{dilog}(e^{\lambda_2 w}) + \frac{1}{6} \pi^2 \\
 &\quad - \frac{1}{2} (\lambda_2 w)^2 - \operatorname{dilog} \left( \frac{1}{1 - e^{\lambda_1 u}} \right) - \ln |e^{\lambda_1 u}| \ln \left| \frac{1}{1 - e^{\lambda_1 u}} \right|
 \end{aligned}
 \tag{22}$$

Equations (21) and (22) may then be used to derive the predicted equation for the amount of failure. The size of the age, the distance traveled, the Weibull and Exponential distribution parameters, and the level of failure are all factors in the predicted equation. The estimated amount of failure will rise in proportion to the worth of life and distance.

### 3.5 Numerical Study

In accordance with Equations (21) and (22) regarding the expected amount of failure and by entering the parameter values and assuming the values of the random variables  $x = u$  and  $y = w$ , the results obtained from the expected amount of failure are as follows:

1. Motorbike with 125 CC

By performing substitution of  $\beta_1 = 1.559355$ ,  $\beta_2 = 1.167246$ ,  $\alpha_1 = 179.132405$ ,  $\alpha_2 = 4020.425463$  parameters to Equation (21) and suppose the value of a random variable is  $x = u$  and  $y = w$ . Hence, Table 6 shows how much failure is predicted to occur.

**Table 6** Expectation of Failure for 125 CC

$u$ (days)	$w$ (km)			
	$0 \leq u \leq 61$	$61 < u \leq 122$	$122 < u \leq 243$	$243 < u \leq 365$
$0 \leq w \leq 1,000$	0.2531	0.5822	0.9599	1.9052
$1,000 < w \leq 4,000$	0.5589	1.0491	2.1512	2.3106
$4,000 < w \leq 8,000$	0.6108	1.2828	3.4769	3.8891
$8,000 < w \leq 12,000$	2.8684	4.7063	5.6332	7.1356

2. Motorbike with 150 CC

By performing substitution of  $\lambda_1 = 0.007282$ ,  $\lambda_2 = 0.000234$  parameters to Equation (22) and suppose the value of a random variable is  $x = u$  and  $y = w$ . Hence, Table 7 shows how much failure is predicted to occur.

**Table 7** Expectation of Failure for 150 CC

$u$ (days)				
$w$ (km)	$0 \leq u \leq 61$	$61 < u \leq 122$	$122 < u \leq 243$	$243 < u \leq 365$
$0 \leq w \leq 1,000$	0.2427	0.2603	0.4676	0.4932
$1,000 < w \leq 4,000$	0.9559	2.3515	2.4306	3.7259
$4,000 < w \leq 8,000$	1.7757	2.9746	5.6336	7.524
$8,000 < w \leq 12,000$	2.6759	5.4013	8.7437	10.4238

3. Motorbike with 160 CC

By performing substitution of  $\lambda_1 = 0.008034$ ,  $\lambda_2 = 0.000816$  parameters to Equation (22) and suppose the value of a random variable is  $x = u$  and  $y = w$ . Hence, Table 8 shows how much failure is predicted to occur.

**Table 8** Expectation of Failure for 160 CC

$u$ (days)				
$w$ (km)	$0 \leq u \leq 61$	$61 < u \leq 122$	$122 < u \leq 243$	$243 < u \leq 365$
$0 \leq w \leq 1,000$	0.7301	1.5707	2.6851	4.3326
$1,000 < w \leq 4,000$	1.7279	2.2129	3.5075	6.9813
$4,000 < w \leq 8,000$	4.4466	5.4188	10.474	19.6457
$8,000 < w \leq 12,000$	6.2872	12.579	21.877	22.986

Tables 6, 7, and 8 present the expected value of the amount of failure for each motorbike. The table displays the  $u$  value range from 0 to 1 year with a 2 to 4-month interval. As for the  $w$  interval between 0 to 12,000 km with an interval of 4,000 km. Based on the results in the table, it is found that the expected number of product defects has increased in proportion to the increasing age and use of motorbikes.

**3.6 Warranty Cost**

Equation (16) may be used to get the optimal cost based on the service cost data for four times as seen in Table 9.

**Table 9** Details on Warranty Cost

Type	Service Cost	Sum	Replacement Costs
125 CC	Rp 9,600,-	Rp 99,200,-	Rp 55,100,-
	Rp 23,200,-		
	Rp 35,200,-		
	Rp 31,200,-		
150 CC	Rp 9,600,-	Rp 99,200,-	Rp 55,100,-
	Rp 23,200,-		
	Rp 35,200,-		
	Rp 31,200,-		
160 CC	Rp 8,400,-	Rp 136,240,-	Rp 60,400,-
	Rp 32,880,-		
	Rp 48,480,-		
	Rp 46,480,-		

The warranty costs are determined as follows by using Equation (15) and the outcomes of the predicted amount of failure for each motorbike.

1. Warranty Cost for 125 CC

**Table 10** Expectation of Warranty Cost for 125 CC

$u$ (days)				
$w$ (km)	$0 \leq u \leq 61$	$61 < u \leq 122$	$122 < u \leq 243$	$243 < u \leq 365$
$0 \leq w \leq 1,000$	Rp 80,212,-	Rp 112,853,-	Rp 150,327,-	Rp 244,095,-
$1,000 < w \leq 4,000$	Rp 110,538,-	Rp 159,174,-	Rp 268,497,-	Rp 284,308,-
$4,000 < w \leq 8,000$	Rp 115,690,-	Rp 182,355,-	Rp 400,010,-	Rp 440,903,-
$8,000 < w \leq 12,000$	Rp 339,640,-	Rp 521,965,-	Rp 613,913,-	Rp 762,953,-

2. Warranty Cost for 150 CC

**Table 11** Expectation of Warranty Cost for 150 CC

$u$ (days)				
$w$ (km)	$0 \leq u \leq 61$	$61 < u \leq 122$	$122 < u \leq 243$	$243 < u \leq 365$
$0 \leq w \leq 1,000$	Rp 79,172,-	Rp 80,925,-	Rp 101,481,-	Rp 104,027,-
$1,000 < w \leq 4,000$	Rp 149,931,-	Rp 288,372,-	Rp 296,212,-	Rp 424,706,-
$4,000 < w \leq 8,000$	Rp 231,248,-	Rp 350,175,-	Rp 613,955,-	Rp 801,481,-
$8,000 < w \leq 12,000$	Rp 320,546,-	Rp 590,913,-	Rp 922,475,-	Rp 1,089,137,-

## 3. Warranty Cost for 160 CC

**Table 12** Expectation of Warranty Cost for 160 CC

$u$ (days)				
$w$ (km)	$0 \leq u \leq 61$	$61 < u \leq 122$	$122 < u \leq 243$	$243 < u \leq 365$
$0 \leq w \leq 1,000$	Rp 159,864,-	Rp 274,388,-	Rp 426,224,-	Rp 650,679,-
$1,000 < w \leq 4,000$	Rp 295,813,-	Rp 361,893,-	Rp 538,262,-	Rp 1,011,528,-
$4,000 < w \leq 8,000$	Rp 666,198,-	Rp 798,662,-	Rp 1,487,378,-	Rp 2,736,926,-
$8,000 < w \leq 12,000$	Rp 916,968,-	Rp 1,774,163,-	Rp 3,040,922,-	Rp 3,192,013,-

The warranty cost for motorcycles with 125 CC, 150 CC, and 160 CC engine capacities are shown in Tables 10, 11, and 12 in the form of a warranty offer with a duration of ( $u$ ) up to 1 year and use of ( $w$ ) up to 12,000 km. The value of  $u$  (time) for a year with intervals of 2 and 4 months is shown in the table. With a 4,000 km interval, the value of  $w$  (distance) is shown up to a maximum of 12,000 km. The anticipated cost of failure to each motorbike has an impact on the warranty charge. The cost of the warranty will rise as estimated failure value rises.

The paper discussed the impact of anticipated failure costs on the initial purchase cost of the motorcycles, which includes the warranty cost paid by the customer when buying the motorcycles. In other words, we suggests that the anticipated cost of potential future failures and warranty-related services is factored into the price that customers pay when purchasing motorcycles. The study might provide the overall concept that as the estimated failure rate or cost increases, the warranty cost also rises, but it does not provide the exact formula or rate of increase.

In practice, the relationship between warranty costs and estimated failure values can vary significantly based on factors such as the type of product, the warranty terms and conditions, the manufacturer's pricing strategy, and the historical data related to warranty claims and failures. It's important for consumers to carefully review the warranty terms and conditions provided by manufacturers before purchasing a motorcycle. While some manufacturers may offer shorter warranty periods, others may provide longer warranties, extended warranties, or additional coverage for specific components or systems. Understanding the warranty terms can help consumers make informed decisions when choosing a motorcycle and assessing the level of protection they'll have in case of unexpected issues.

## 4 Conclusion

This paper has a contribution in determining the warranty cost based on estimated failure in two dimensions. A warranty is a post-transaction service between seller and consumer that is provided for the use of goods on an ongoing basis. Giving guarantees is considered to be able to attract consumers because the company will provide the right to protection or guarantees against product dissatisfaction received by consumers. The provision of guarantees is closely related to the analysis of the survival of a product until it fails. If the particular distribution of data is known, then the survival of a population or sample that is observed can be obtained by estimating the parameters of the distribution. Warranty costs must be calculated so as not to harm or exceed the predicted failure that will arise. The failure expectation model for two-dimensional policy guarantees through the Bayesian approach is more optimal to use in estimating parameters. The warranty policy offered by manufacturers for new motorcycles can vary significantly from one manufacturer to another and may depend on factors such as the type of motorcycle, the intended market, and the manufacturer's business strategy. Manufacturers often design warranty policies to align with their marketing and competitive goals.

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