Comparing Classical and Sequential Parameter Estimation in Power Rayleigh Distribution

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Abstract

This study explores the novel application of the performance characteristics of sequential probability ratio test (SPRT) for estimation the scale parameter (θ) of the power Rayleigh distribution, focusing on its scale parameters (θ) that are the fundamental to defining its properties and shaping the statistical characteristics. For testing the null hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, a stop rule method is utilized to optimize the sample size. Through this approach, we derive approximations for key performances metrics, including the operating characteristic (OC) function and the average sample number (ASN). The study also confirms the existence of the function $L(\theta)$, necessary to meet the ASN requirements in sequential testing. The new method is comprised of the efficiency of the SPRT compared to the classical Neyman-Pearson (N-P) technique. A comprehensive simulation study shows that the proposed method of SPRT significantly reduces the sample size required for the hypothesis testing by **65.99%** to **77.49%** compared to the Neyman-Pearson (N-P) test without compromising accuracy.

Keywords: Sequential probability ratio test, operating characteristics, average sample number, acceptance and rejection regions, power Rayleigh distribution, Neyman-Pearson procedure, Type I error and Type II error.

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1 Introduction

The original purpose of sequential analysis, firstly developed by Wald (1947), was to improve quality control during World War II. This goal was accomplished to such an extent that it was instantly classified. Sequential analysis tries to minimize the sample size while maintaining decision-making ability by treating it as a random variable for that specific method. This is achieved by stopping data collecting after one more observation has been made in order to determine whether a choice can be made. There are several uses for the potential to lower the sample size needed to decide on an experiment because this methodology conserves resources and makes financing more easily obtained.

Since, Wald (1947), numerous sequential tests have been suggested to evaluate various hypotheses. For instance, in the books Whitehead and Jones (1983) and Jennison and Turnbull (2000) offered various applications of sequential and group sequential method-based methodologies to the design and implementation of clinical trials. For further information, readers can referred to their books.

Darkhovsky (2011) proposed a sequential approach that minimizes the Bayesian risk maximum over a variety of prior parameter distributions in order to study the problem of sequential testing for two composite hypotheses. Tartakovsky et al. (2015) went into great detail about current advancements by developing the systematic hypothesis testing and change point detection in the theory of sequential analysis in their book . Kharin (2016) looked on the reliability and efficiency of sequential testing. Nakamura et al. (2016) suggested a sequential test that find the lowest dose that has a threshold effect.

A universal sequential test was developed by Li et al. (2017) to identify outliers in all of the gathered observation sequences. An ideal truncated group sequential test for binomial proportions was proposed by Hu and Wang (2018). The study of Dyrssen and Ekstrom (2018) examined the sequential testing of two simple hypothesis for the shift of a Brownian motion with expensive observations. Zou et al. (2019) presented a nonparametric sequential test based on empirical likelihood to assess for treatment effects.

Likelihood ratio test method is widely applicable method for the two simple/composite hypothesis with a fixed sample size. For example, Self and Liang (1987) obtained the asymptotic distribution of maximum likelihood ratio statistics under nonstandard conditions. Fan and Zhang (2004) gave the empirical likelihood ratio test for non-parametric functions. Ferrari and

Cysneiros (2008) applied the modified likelihood ratio method to study the exponential family nonlinear model and get the modified likelihood ratio statistics. Giampaoli and Singer (2009) used likelihood ratio method to estimated the variance parameter of the linear mixed model. Huang et al. (2013) used the Lq-likelihood ratio method for the testing of shape parameter of the generalized extremum distribution. Similar method is propsed by Qin and Priebe (2017) for the general pollution distribution also obtained the asymptotic distribution of Lq-likelihood ratio test statistics.

The presented work shows the importance of the sequential testing procedure compared with the classical fixed-sample methods, as SPRT offers a more flexible approach. In this case, the power Rayleigh distribution which is applied in survival studies, reliability analysis, and communication systems, SPRT has shown better performance than classical procedures due to its faster convergence and lower average sample size. Therefore, the comparison between SPRT and classical methods under the power Rayleigh distribution is of considerable practical and theoretical importance in statistical analysis and real-time applications.

2 SPRT Procedure

Wald (1947) introduced a extension of the sequential estimation by introducing the sequential probability ratio test (SPRT). Consider a sequence of i.i.d random variable $X_1, X_2, \ldots$ having the common probability density function (p.d.f) $f(x|\theta)$, $\theta \in \Theta$. To testing the one simple null hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$, where $\theta_0 \neq \theta_1$ are two specified numbers in the parameter space Θ. We also have two preassigned numbers $0 < \alpha, \beta < 1$, and corresponding to type I and type II error probabilities.

Let $X_n = (X_1, X_2, \ldots X_n)$, when θ is true parameter value, the likelihood function of X_n is

$$L(X_n, \theta) = \Pi_{i=1}^{n} f(X_i, \theta)$$

We denote $L_0(i) = \Pi_{i=1}^{n} f(X_i, \theta_0)$ and $L_1(i) = \Pi_{i=1}^{n} f(X_i, \theta_1)$, respectively under H_0 and H_1. The most powerful (MP) level α test has the form

$$\text{Reject } H_0 \text{ if and only if } \quad \frac{L_1(i)}{L_0(i)} \geq k \tag{1}$$

where $k > 0$ is to be determined appropriately.

The test (1) is the best among all fixed sample size (=i) tests at level α. But its type II error probability can be quite bit larger than β. In order to meet both (α, β) requirements. Abraham Wald (1940) looked at this problem from a different angle. Having observed X_n let us consider the sequence of successive likelihood ratios is defined as:

$$\Lambda_i = \frac{L_1(i)}{L_0(i)} \quad i = 1, 2, 3...$$

In other words, we observe $X_1, X_2, X_3, \ldots$ one by one and accordingly obtain $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ in a sequential manner. As soon as we find one Λ_j in the sequence that appears **too small or too large** we quit sampling, and we decide in favor of $H_0(orH_1)$ whenever Λ_i is too small (or too large), i $= 1, 2, \ldots.$

By choosing two constants A and B such that $0 < B < A < \infty$, sequential probability ratio test (SPRT) is then implemented as –

We continue observing $X_1, X_2, \ldots$ one by one in a sequence until the likelihood ratio Λ_i goes out of the interval (B, A) for the first time then accept H_0 if $\Lambda_i \leq B$, reject H_0 if $\Lambda_i \geq A$ and continue sampling if $B < \Lambda_i < A$ for $i = 1, 2, \ldots n$ number of experiments. The constants A and B are computed using the values of Type I and Type II error probabilities α and β. The mathematical formula of A and B is defined as $A = (1 - \beta)/\alpha$ and $B = \beta/(1 - \alpha)$. The performance of SPRT can be measured by the two criteria one is Operating Characteristic (OC) fuction and another is Average sample number (ASN).

Operating Characteristic (OC) fuction denotes the probability that the sequential test will lead to the acceptance of the hypothesis $H_0 : \theta = \theta_0$ when θ is the true value. It is denoted as $L(\theta)$ and relates how well the test procedures achieves its objective of making correct decisions. The number of observation N required by a sequential procedure to reach a decision is not predetermined but a random variable. If we repeat the same sequential then we shall get different values N. Small value of N on an average is preferable. This average value of N is called the Average Sample Number (ASN). Wald (1947) suggested the mathematical formula for the OC and ASN as –

$$L(\theta) = \frac{A^h - 1}{A^h - B^h} \tag{2}$$

where A and B are constants, depends on the Type I and Type II error of probabilities both, and 'h' is the solution of non-zero equation and is given as

$$E[e^{Z_i}]^h = 1 \tag{3}$$

the ASN function under H_0 and H_1 is given by

$$\begin{aligned} E_0(N) =& \frac{(1-\alpha)\ln B + \alpha \ln A}{E(Z_i)} \\ E_1(N) =& \frac{\beta \ln B + (1-\beta)\ln A}{E(Z_i)} \end{aligned} \tag{4}$$

where, $Z_i = log(f(X_n, \theta_1)/f(X_i, \theta_0))$ provided $E(Z_i) \neq 0$.

SPRT is applied by various author, one may referred to Chaturvedi et al.(2000), Sevil and Demirhan (2008), Pandit and Gudaganavar (2010), Kharin (2016), Ning and Opperman (2019) and Surinder.K et al. (2022).

The acceptance and rejection regions are crucial when statistical decision under SPRT are depends on the data. Similar to classical theory, rejection region indicate strong evidence against the null hypothesis and acceptance region signifies the values for which the null hypothesis can not be rejected.

Rayleigh (1880) introduced a distribution, named Rayleigh distribution with many applications in the field of engineering and medical disciplines for data modeling. It is often applied in hydrology to model phenomena that involves the magnitude of extreme or varying value, such as wave heights, riven flow velocity and wind speeds over water bodies. To enhance the flexibility of Rayleigh model Power transformation $x = y^{\frac{1}{\nu}}$ is used. The cumulative distribution function (CDF) of the X is presented by

$$F(x) = P(X \leq x) = P(Y \leq x^{\nu}) = F_y(x^{\nu})$$

Thus, the probability density function (PDF) and cumulative density function (CDF) of power Rayleigh (PR) distribution is given by –

$$f(x) = \frac{\nu}{\theta^2} x^{2\nu-1} e^{\frac{-x^{2\nu}}{2\theta^2}} \quad x > 0, \theta, \nu > 0, \quad F(x) = 1 - e^{\frac{-x^{2\nu}}{2\theta^2}} \tag{5}$$

where, θ is scale parameter and ν is shape parameter. The presented study deal with the estimation of scale parameter of the power Rayleigh distribution.

3 Sequential Probability Ratio Test (SPRT) for Power Rayleigh Distribution

Suppose that $X_1, X_2, X_3, \ldots$ be an independent and identically distributed random sample from the pdf (5). Here, testing the null hypothesis $H_0 : \theta =$

θ_0 $vs.$ an alternative $H_1 : \theta = \theta_1$.The test statistic for the test SPRT is defined as –

$$\Lambda_n = \left(\frac{\theta_0}{\theta_1}\right)^2 exp\left[-\frac{x^{2\nu}}{2}\left(\frac{1}{\theta_0^2}-\frac{1}{\theta_1^2}\right)\right]$$

$$\ln\ \Lambda_n = 2\ln\left(\frac{\theta_0}{\theta_1}\right)+\frac{x^{2\nu}}{2}\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right) \tag{6}$$

The decision rule is defined as –

1. Accept H_0 if $\Lambda_n = \left(\frac{\theta_0}{\theta_1}\right)^2 exp\left[-\frac{x^{2\nu}}{2}\left(\frac{1}{\theta_0^2}-\frac{1}{\theta_1^2}\right)\right] \leq B$
2. Reject H_0 if $\Lambda_n = \left(\frac{\theta_0}{\theta_1}\right)^2 exp\left[-\frac{x^{2\nu}}{2}\left(\frac{1}{\theta_0^2}-\frac{1}{\theta_1^2}\right)\right] \geq A$
3. Continue taking sampling if $B < \left(\frac{\theta_0}{\theta_1}\right)^2 exp\left[-\frac{x^{2\nu}}{2}\left(\frac{1}{\theta_0^2}-\frac{1}{\theta_1^2}\right)\right] < A$

4 Operating Characteristic (OC) for SPRT

As, from the decision rule, SPRT method depends on the pre-defined constants A and B. So, by selecting A and B, we are studying the properties of SPRT that can be turned into the study of random walks. In this Section, some characteristic of SPRT are derived for the pdf of (5).

Let to test $H_0 : \theta = \theta_0$ against the hypothesis $H_1 : \theta = \theta_1$, with the given constants α and β, suppose ν is known. To find the OC function we obtained the value of Z_i as

$$\begin{aligned} Z_i &= \ln\left(\frac{f(X_i,\theta_1)}{f(X_i,\theta_0)}\right) \\ &= \ln\left(\frac{\theta_0}{\theta_1}\right)^2 \exp\left[-\frac{x^{2\nu}}{2}\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right)\right] \\ &= 2\ln\frac{\theta_0}{\theta_1} - \frac{x^{2\nu}}{2}\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right) \end{aligned} \tag{7}$$

From the equations, (3) and (7), we have

$$E\left[e^{Z_i}\right]^h = \frac{\left(\frac{\theta_0}{\theta_1}\right)^{2h}}{\left[h\theta^2\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right)+1\right]} \tag{8}$$

To solve this equation, taking logarithm and using the expression $\ln(1+x); -1 < x < 1$ in (3.4). Expanding the terms up to third degree we got the quadratic equation in 'h' –

$$\frac{h^2}{3}\left[\theta^2\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right)\right]^3-\frac{h}{2}\left[\theta^2\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right)\right]^2$$
$$+\theta^2\left(\frac{1}{\theta_1^2}-\frac{1}{\theta_0^2}\right)-2\ln\left(\frac{\theta_0}{\theta_1}\right)=0 \qquad (9)$$

After solving the (9), numerical values of OC function under the scale parameter of power Rayleigh distribution is obtained. For testing $H_0 : \theta = 12$ vs $H_1 : \theta = 14$ with $\alpha = \beta = 0.05$.The numerical values of OC functions are represented in Table 1 and curve are plotted in Figure 1. The resulted numerical values at $\theta = 12$ and $\theta = 14$ are near to and 0.95 and 0.05 are satisfactory.

Table 1 Numerical value for OC function under $H_0 : \theta = 12$ vs $H_1 : \theta = 14$ with $\alpha = \beta = 0.05$

θ	$L(\theta)$	θ	$L(\theta)$
11.0	0.999	12.7	0.67
11.1	0.999	12.8	0.6
11.2	0.998	12.9	0.527
11.3	0.997	13.0	0.453
11.4	0.996	13.1	0.382
11.5	0.993	13.2	0.317
11.6	0.99	13.3	0.258
11.7	0.985	13.4	0.206
11.8	0.978	13.5	0.163
11.9	0.968	13.6	0.126
12.0	**0.954**	13.7	0.096
12.1	0.936	13.8	0.082
12.2	0.911	13.9	0.063
12.3	0.88	14.0	**0.057**
12.4	0.84	14.1	0.025
12.5	0.792	14.2	0.016
12.6	0.735	14.3	0.008

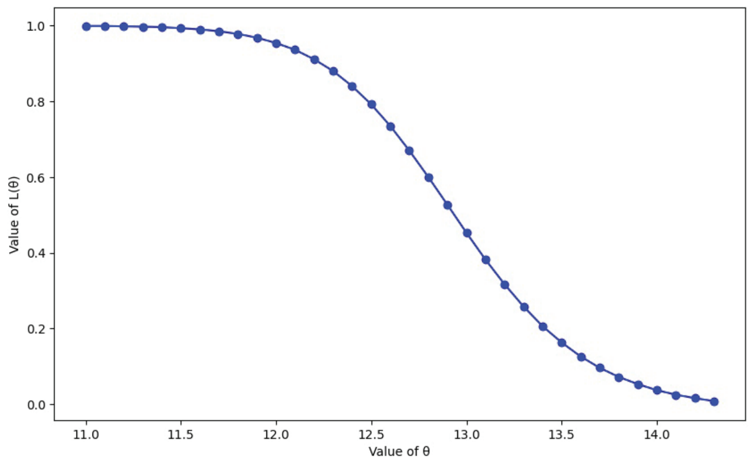

Figure 1 OC function curve for Power Rayleigh distribution.

5 Average Sample Number (ASN) for the SPRT

In a sequential estimation procedure, the sample size required to make a decision is denoted by N. The true distribution of the observations made throughout the sampling procedure determines how this random variable will be distributed. The function of ASN, approximately represented as

$$E(N|\theta) = \frac{L(\theta)\ln B + [1 - L(\theta)]\ln A}{E(Z)} \tag{10}$$

also given that $E(Z) \neq 0$, where

$$E(Z) = 2\ln\left(\frac{\theta_0}{\theta_1}\right) - \theta^2\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right) \tag{11}$$

where, $E[x^{2\nu}] = 2\theta^2$ (Mahmoud, et al., 2020).

From Equation (4), the function of ASN under H_0 and H_1 is represented as

$$E_0(N) = \frac{(1-\alpha)\ln B + \alpha\ln A}{2\ln\left(\frac{\theta_0}{\theta_1}\right) - \theta_0^2\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)} \tag{12}$$

and

$$E_1(N) = \frac{\beta\ln B + (1-\beta)\ln A}{2\ln\left(\frac{\theta_0}{\theta_1}\right) - \theta_1^2\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)} \tag{13}$$

After solving the (13), numerical values of ASN function under the scale parameter of power Rayleigh distribution is obtained. For testing $H_0 : \theta = 12$ vs $H_1 : \theta = 14$ with $\alpha = \beta = 0.05$, the numerical values of ASN functions (10) are represented in Table 2 and curve are plotted in Figure 2. The ASN curve is a plotted against the values of θ. The Table and Curve gives satisfactorily results.

The ASN is maximized for intermediate values of θ because in the midrange, the observations would support neither hypothesis with strong confidence, which would increase the number of observations needed to make a decision to accept or reject the hypothesis. If θ is near θ_0 and near θ_1, on the other hand, the generated data bolster one alternative hypothesis, causing the test statistic to cross the respective boundaries earlier than it would if θ were different, and there will be fewer average sample sizes. Therefore, the greater the difficulty in distinguishing between the hypotheses, the larger the ASN, and the closer it comes to the parameter values that correspond to making better decisions, the smaller it becomes.

Table 2 Numerical value for ASN function under $H_0 : \theta = 12$ vs $H_1 : \theta = 14$ with $\alpha = \beta = 0.05$

θ	E(N)	θ	E(N)
11.0	34.446	12.7	89.970
11.1	36.142	12.8	91.337
11.2	38.016	12.9	91.470
11.3	40.092	13.0	90.363
11.4	42.395	13.1	88.126
11.5	44.951	13.2	84.958
11.6	47.787	13.3	81.106
11.7	50.924	13.4	76.819
11.8	54.380	13.5	72.326
11.9	58.156	13.6	67.808
12.0	62.239	13.7	63.401
12.1	66.581	13.8	59.195
12.2	71.097	13.9	55.246
12.3	75.656	14.0	51.580
12.4	80.068	14.1	48.205
12.5	84.100	14.2	45.120
12.6	87.487	14.3	42.332

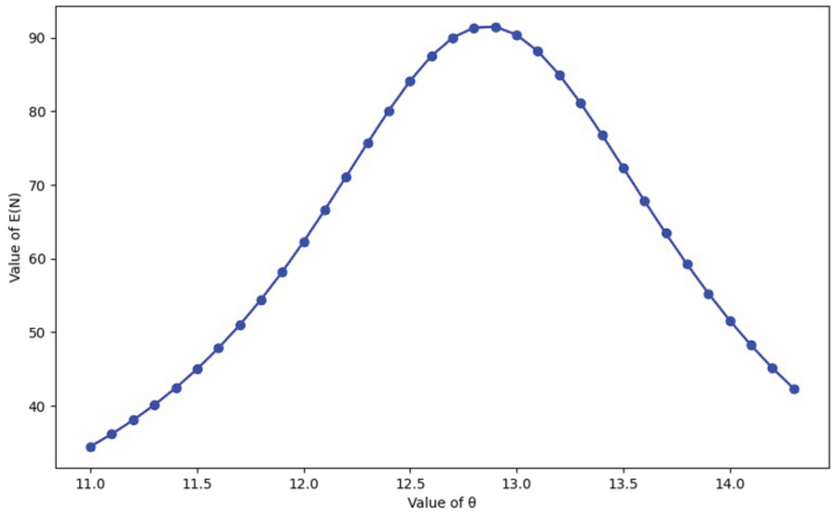

Figure 2 ASN function curve for Power Rayleigh distribution.

6 Acceptance and Rejection region

Here, we define, $Y(n) = \sum_{i=1}^{n} X_i^{2\nu}$ and N=first integer $n(\geq 1)$, for which the inequality $Y(n) \leq c_1 + dn$ or $Y(n) \geq c_2 + dn$ holds with the constants

$$c_1 = \frac{\ln B}{\frac{1}{2}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)}, \quad c_2 = \frac{\ln A}{\frac{1}{2}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)} \quad \text{and} \quad d = \frac{2\ \ln\frac{\theta_0}{\theta_1}}{\frac{1}{2}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)} \tag{14}$$

The decision rule for the SPRT is

1. when $Y_n \geq c_1 +$ dn, we have to stop the experiment and accept H_0
2. when $Y_n \leq c_1 +$ dn, we have to stop the experiment and reject H_0

Remarks: The Figure 3, shows the acceptance and rejection regions for H_0 under the case when $H_0 : \theta_0 = 12$ vs $H_1 : \theta_1 = 14$ for $\alpha = \beta = 0.05$. The values of constants $c_1 = 494.6657$, $c_2 = -494.6657$ and $d = 51.79463$, respectively. Thus, if $Y(N) \leq 494.6657N + 51.79463$, we accept H_0 and if $Y(N) \geq -494.6657N + 51.79463$, we accept H_1. At the intermediate stages, we continue sampling.

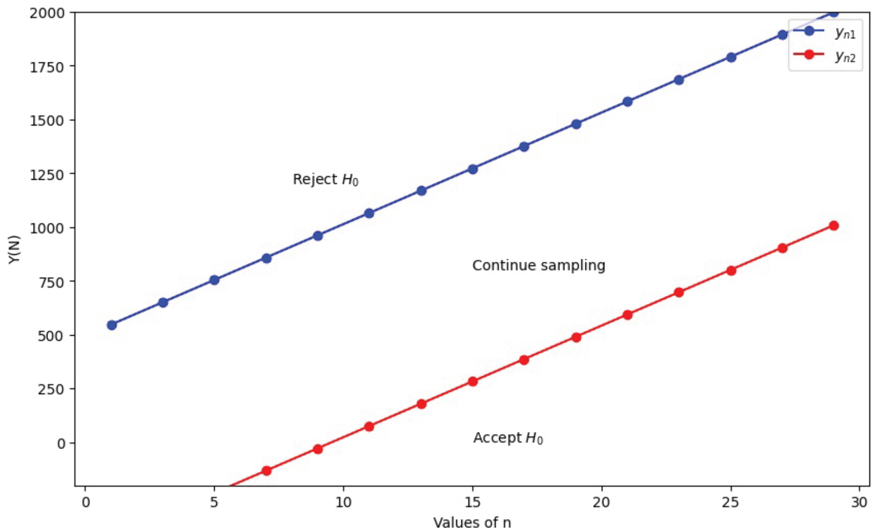

Figure 3 Acceptance and Rejection region.

7 Fixed-Sample Size Procedure

Let $\{X_i\}$ be i.i.d. having the pdf (5). We are interested to $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$. For the estimation of fixed-sample size we are using Neyman-Pearson test method. As per N-P method, the optimal fixed quantity is defined as where, n and K satisfy

$$\begin{aligned} P_{\theta_0}(\bar{X} \geq K) &= \alpha \\ P_{\theta_1}(\bar{X} < K) &= \beta \end{aligned} \tag{15}$$

So, as per the central limit theorem, the distribution of $\bar{X}$ is,

$$\frac{\sqrt(n)(\bar{X} - E[X_i])}{\sqrt{(var(X_i))}} \sim AN(0,1) \tag{16}$$

So, we have from the above equation as

$$\begin{aligned} P_{\theta_0}\left(\frac{\sqrt{(n)}(\bar{X} - E_{\theta_0}[X_i])}{\sqrt{(var_{\theta_0}(X_i))}} \geq \frac{\sqrt{(n)}(K - E_{\theta_0}[X_i])}{\sqrt{(var_{\theta_0}(X_i))}}\right) &= \alpha \\ P_{\theta_1}\left(\frac{\sqrt{(n)}(\bar{X} - E_{\theta_1}[X_i])}{\sqrt{(var_{\theta_1}(X_i))}} < \frac{\sqrt{(n)}(K - E_{\theta_1}[X_i])}{\sqrt{(var_{\theta_1}(X_i))}}\right) &= \beta \end{aligned} \tag{17}$$

Table 3 Sample size under N-P test

	α			
β	0.01	0.03	0.05	0.1
0.01	29.523	24.356	21.817	18.184
0.03	23.936	19.307	17.054	13.862
0.05	16.524	16.868	14.766	11.807
0.1	15.996	13.437	11.569	8.969

That is,

$$Z_\alpha = \frac{\sqrt{(n)}(K - E_{\theta_0}[X_i])}{\sqrt{(var_{\theta_0}(X_i))}}$$
$$Z_{1-\beta} = \frac{\sqrt{(n)}(K - E_{\theta_1}[X_i])}{\sqrt{(var_{\theta_1}(X_i))}} \tag{18}$$

where, Z_α and $Z_{1-\beta}$ are quantiles of the standard normal distribution.

For testing $H_0 : \theta_0 = 12$ against the alternative $H_1 : \theta_1 = 14$, $\nu = 1$ for various significance levels (α) and power values $(1-\beta)$, Table 3 showing the sample sizes required for a Neyman-Pearson (N-P) test. The presented table helps determine the Sample size required to achieve a balance between Type I and Type II errors in hypothesis testing, with smaller values of (α) and β requiring larger sample sizes to maintain the power of the test.

8 The Efficiency of the SPRT

Let $\{X_i\}$ be iid with pdf $f(x, \theta)$. If one is interseted to the simple null hypothesis Vs the simple alternative, subject to prescribed error probabilities α and β. One can examine the effectiveness of the SPRT in relation to any procedure D with the optimum fixed sample size $N_0(\alpha, \beta)$. Then a test may be regarded as preferable to another test of the class if the former requires a smaller number of observations, on the average, than the latter. So, the efficiency of a given test is defined as the ratio

$$R_e(\theta_0) = \frac{N_0(\alpha, \beta)}{E_{\theta_0}(N)}$$
$$R_e(\theta_1) = \frac{N_0(\alpha, \beta)}{E_{\theta_1}(N)} \tag{19}$$

under H_0 and under H_1 respectively.

The percentage of saving by the SPRT is obtained by the formula given by Wald (1947) as

$$R_e(\theta_0)\% = \left[\frac{R_e(\theta_0) - 1}{R_e(\theta_0)}\right] * 100$$

$$R_e(\theta_1)\% = \left[\frac{R_e(\theta_1) - 1}{R_e(\theta_1)}\right] * 100 \tag{20}$$

under H_0 and under H_1, respectively.

The ratio of sample size required by SPRT and N-P test is defined as

$$r = \frac{E_{\theta_0}(N) + E_{\theta_1}(N)}{2n} \tag{21}$$

9 Simulation Study

In this section, to confirm the superiority of the SPRT approach, we will compare the average sample sizes produced by the Neyman-Pearson (N-P) test procedure and the sequential method of estimation. R software is used for the simulation.

Let $\theta_0 = 12$ and $\theta_1 = 14$ represent parameter values chosen for a statistical model. These values are considered optimal because they yield the maximum efficiency under $\alpha = 0.01, 0.03, 0.05, 0.1$ and $\beta = 0.01, 0.03, 0.05, 0.1$. The numerical value of the ASN required by SPRT method at the given tolerance level are presented in Table 4. It is evident that as the value of α and β both increase, the expected sample size required for both $E_{\theta_0}(N)$ and $E_{\theta_1}(N)$ decreases, this pattern that higher error tolerance (larger α or β) lead to smaller sample size. Their corresponding efficiency and percentage of savings by SPRT are represented in Tables 5 and 6, respectively.

Table 4 ASN Functions under various value of α and β

	$E_{\theta_0}(N)$					$E_{\theta_1}(N)$			
	α					α			
β	0.01	0.03	0.05	0.1	β	0.01	0.03	0.05	0.1
0.01	7.851	7.553	7.282	6.661	0.01	6.727	6.472	6.24	5.707
0.03	5.955	5.696	5.464	4.94	0.03	5.102	4.881	4.682	4.233
0.05	5.074	4.834	4.62	4.143	0.05	4.348	4.142	3.959	3.55
0.1	3.878	3.664	3.477	3.064	0.1	3.323	3.14	2.979	2.626

Table 5 The efficiency of the SPRT relative to the fixed sample size procedure

	$R_e(\theta_0)$					$R_e(\theta_1)$			
	α					α			
β	0.01	0.03	0.05	0.1	β	0.01	0.03	0.05	0.1
0.01	3.760	3.224	2.996	**2.729**	0.01	4.388	3.763	3.496	3.186
0.03	4.0195	3.3896	3.1212	2.8061	0.03	4.6915	3.9556	3.6425	3.2747
0.05	3.2566	3.4894	3.1961	2.8499	0.05	3.8004	4.0724	3.7297	3.3259
0.1	4.1248	3.6673	3.3273	2.9272	0.1	**4.8137**	4.2793	3.8835	3.4155

Table 6 Percentage of savings by the SPRT

	$R_e(\theta_0)\%$					$R_e(\theta_1)\%$			
	α					α			
β	0.01	0.03	0.05	0.1	β	0.01	0.03	0.05	0.1
0.01	0.7341	0.6899	0.6662	0.6337	0.01	0.7721	0.7343	0.714	0.6862
0.03	0.7512	0.705	0.6796	0.6436	0.03	0.7868	0.7472	0.7255	0.6946
0.05	0.6929	0.7134	0.6871	0.6491	0.05	0.7369	0.7544	0.7319	0.6993
0.1	0.7576	0.7273	0.6995	0.6584	0.1	0.7923	0.7663	0.7425	0.7072

Table 7 Efficiency under r

	α			
β	0.01	0.03	0.05	0.1
0.01	4.0503	3.4732	3.2269	2.9405
0.03	4.3296	3.6508	3.3617	3.0223
0.05	3.5075	3.7585	3.4424	3.0695
0.1	4.4427	3.9497	3.584	3.1525

Table 8 Percentage of savings by the SPRT under ratio r

	α			
β	0.01	0.03	0.05	0.1
0.01	75.31	71.21	69.01	**65.99**
0.03	76.9	72.61	70.25	66.91
0.05	71.49	73.39	70.95	67.42
0.1	**77.49**	74.68	72.10	68.28

The ratio of sample size required by SPRT and N-P test and corresponding percentage of savings by the SPRT under $\theta_0 = 12$ and $\theta_1 = 14$ is showing by the Tables 7 and 8, respectively.

10 Conclusion

The Table 4 provides insights into how the average sample number (ASN) changes with different Type I and Type II error rates in hypothesis testing, under both null and alternative hypothesis scenarios. Table 5 compares the efficiency of the Sequential Probability Ratio Test (SPRT) with the Neyman-Pearson (N-P) test under different combinations of α and β. In general, the efficiency values are between 2.729 and 4.814, meaning that in the cases considered, the SPRT has an efficiency between 2.729 and 4.814 of the Neyman-Pearson test.

Table 6, suggests that the efficiency of the SPRT procedure varies with different levels of α and β. Higher value in the table indicate higher relative efficiency of SPRT over fixed sample size procedure for the given hypothesis θ_0 or θ_1. It is also evident that lower values of α and β lead to higher efficiency in both cases, through there are slight variations across different parameter combinations. Practically, these results indicate that the SPRT can reach reliable statistical decisions using considerably fewer observations, leading to reduced experimental cost, shorter testing duration, and lower data collection effort. Such improvements are particularly valuable in reliability analysis and life-testing applications involving the Power Rayleigh distribution, where obtaining observations may be time-consuming or expensive.

As demonstrated from the Table 7 the combined efficiency (r) under H_0 and H_1 is represented for various values of Type I and Type II errors. From Table 8, it is evident that ASN saves 65.99% to 77.49% percentage as compared to classical approach. Observations reveal that the sample size needed by the SPRT technique is significantly less than the ideal the sample size needed to perform the Neyman-Pearson test, which further demonstrates the benefits of the SPRT approach.

A limitation in comparing the classical test with the Sequential Probability Ratio Test (SPRT) for the Power Rayleigh distribution is that the two tests are based on different frameworks and stopping rules. Direct comparisons of efficiency and performance are not as easy to make in the classical test, since the sample size is fixed; in the SPRT, the sample size is variable and depends on the data observed. Moreover, the performance of the SPRT is very much dependent on the values of the design parameters θ_1, θ_0 and the desired values of Type I and Type II error probabilities. Additionally, differences in the assumptions of independence and the correct model specification of the Power Rayleigh distribution could impact the comparison, as these assumptions impact the accuracy and robustness of both approaches differently.

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Biography

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