
Fixed Point Theory in Bi-Operational Metric Spaces with Independent Contraction Operations

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Abstract

The Banach contraction principle forms an integral part of nonlinear analysis and is of extensive applicability in fixed point theory and differential equations. Recently, operational metric spaces have been proposed, where in the triangle inequality, the additive structure was replaced by a binary operation $\oplus$. But in previous literature, the choice of the binary operation ($\oplus$) is highly restricted (for example, either addition or max), with the implicit assumption of a zero element identity. In this paper, we introduce a completely novel concept of a bi-operational metric space with two unrelated binary operations, namely, $\oplus$, which controls the triangle inequality-type property, and $\odot$, controlling the process of contraction. In case the conditions of monotonicity and continuity are satisfied in the usual sense, we prove a Banach-like fixed point theorem in the present context. The model does not involve any notion of scalar multiplication whatsoever. Further, we establish the general applicability of our method in proving that the classical metric

space, multiplicative metric space, and ultrametric space are special cases. An example is illustrated on $[2, 3]$ with the operations $a \oplus b = (a - 1)(b - 1) + 1$ and $a \odot b = (a - 1)^{\ln(b-1)} + 1$, for which the inequality $\mathcal{B}(T\xi, Tb) \leq k \odot \mathcal{B}(\xi, b)$ leads to a unique fixed point.

Keywords: Fixed point, Banach contraction, operational metric, bi-operational metric..

1 Introduction

The contraction mapping theorem proposed by Stefan Banach in 1922 [3] indicates that a contraction self-map on a complete metric space possesses only one fixed point. Over time since it was established, this theorem has taken center stage in the study of nonlinear analysis and has been widely used in diverse fields like differential and integral equations. Starting from the classic result of Banach on the contraction principle [3], there have been many generalizations of the notion of metric spaces that aim to enlarge the applicability of fixed point theory. Among some of the examples are b -metric spaces [2, 11] that generalize the triangular inequality via multiplication by a positive constant and multiplicative metric spaces [10] where the sum is replaced by multiplication in the metric relation, and also ultrametric spaces [16] where the triangle inequality is strengthened to $d(\xi, b) \leq \max\{d(\xi, z), d(z, b)\}$.

It is worth describing the b -metric setting a little more precisely, since it is the closest classical antecedent of the idea of altering the triangle inequality itself. Bakhtin [2] and Czerwik [11] retain the additive form of the triangle inequality but weaken it by a fixed multiplicative constant $s \geq 1$,

$$d(\xi, b) \leq s [d(\xi, z) + d(z, b)], \quad \xi, b, z \in \mathcal{O}.$$

The constant s measures how far the space is from being metric, $s = 1$ returns the classical case, while larger values of s permit distance functions, such as $d(\xi, b) = |\xi - b|^p$ with $p > 1$, that violate the ordinary triangle inequality. What is significant for the present work is that the relaxation is still expressed in terms of ordinary addition and ordinary real multiplication; the algebraic operation governing the triangle inequality is never itself replaced. The same observation applies to cone metric spaces [14], where the scalar codomain $[0, \infty)$ is replaced by a cone in an ordered Banach space: the order becomes partial rather than total, but the operation governing the triangle inequality is

still vector addition, with its identity 0. For a detailed account of how fixed point theory developed in the b -metric setting, and of the role played by the constant s , we refer to the survey of Berinde and Păcurar [15]. The framework of the present paper departs from this tradition by replacing the operation itself, and, more importantly, by decoupling the operation that governs the triangle inequality from the one that governs the contraction.

The study of generalized distance spaces has been gaining much importance recently. For example, the work of Jleli and Samet [4] presented an approach unifying different distance spaces such as metric and (b) -metric spaces and also providing some fixed point theorems related to these spaces. Further, the idea of generalized Ω -distance mappings was developed by Abo-dayeh et al. [5] with several fixed point theorems, which have been improved by Abu-Irwaq et al. [6]. Further advancements on generalized distance spaces have been made in [7, 8], and the idea of gamma-distance mappings and its applications has been studied in [9].

The interest in contraction-type principles is not confined to pure analysis. Iterative and recursive schemes are standard tools in reliability and stochastic modelling, and their solvability is often a fixed point statement in disguise. Etesam and Mohtashami Borzadaran [12] apply progressive-batching L-BFGS iterations, with and without momentum, to reliability optimization problems under uncertainty. Parmender et al. [13] analyse a repairable standby system by the regenerative point technique, which expresses reliability measures such as the mean time to system failure and the steady-state availability as solutions of Markov renewal equations of convolution type; that such equations admit exactly one solution is a classical application of the contraction principle on a complete space of functions. In both examples the argument is carried out in an ordinary metric setting. The present paper asks what survives of such arguments when the distance itself takes values in a set whose natural algebraic structure is not additive.

More recently, Adewale et al. [1] have introduced the notion of operational metric spaces by replacing the additive operator in the triangular inequality with a binary operation. Even though this construction provides an interesting algebraic point of view, the contraction conditions are very close to classical notions. We recall the definition of an operational metric space as introduced by Adewale et al. [1].

Definition 1.1. [1] Let $\mathcal{O}$ be a nonempty set and let $\oplus$ be a binary operation on $[e, \infty)$ having identity element e . A mapping $b : \mathcal{O} \times \mathcal{O} \rightarrow [e, \infty)$ is called an operational metric if

- (OM1) $b(\xi, \flat) \geq e$;
 (OM2) $b(\xi, \flat) = e \iff \xi = \flat$;
 (OM3) $b(\xi, \flat) = b(\flat, \xi)$;
 (OM4) $b(\xi, \flat) \leq b(\xi, z) \oplus b(z, \flat)$ for all $\xi, \flat, z \in \mathcal{O}$.

The triple $(\mathcal{O}, b, \oplus)$ is called an operational metric space.

Remark 1.2. [1] If $a \oplus b = a + b$, then one recovers the usual metric space with $e = 0$. If $a \oplus b = \max\{a, b\}$, one obtains an ultrametric-type structure.

The fundamental idea of Adewale et al. [1] is to replace the addition in the triangle inequality with a binary operation $\oplus$

$$b(\xi, \flat) \leq b(\xi, z) \oplus b(z, \flat).$$

The contraction used in [1] is

$$b(T\xi, T\flat) \leq k \cdot b(\xi, \flat), \quad 0 \leq k < 1.$$

Several fixed point theorems were established for specific choices of $\oplus$, particularly $\oplus = +$ and $\oplus = \max$.

Limitations of the Adewale et al. Approach

We identify three fundamental limitations:

1. **The Codomain Issue:** The contraction $b(T\xi, T\flat) \leq k \cdot b(\xi, \flat)$ with $k < 1$ is only valid when $e = 0$. If $e > 0$ and $\xi = \flat$, then $b(\xi, \flat) = e$, and we require $k \cdot e \geq e$, which implies $k \geq 1$, contradicting $k < 1$.
2. **The Scalar Multiplication Issue:** The expression $k \cdot b(\xi, \flat)$ uses ordinary real multiplication, which is not part of the operational metric structure. The contraction mixes two different operations.
3. **No Genuine Generalization:** When $\oplus = +$, the theory reduces to Banach's theorem. When $\oplus = \max$, it reduces to ultrametric-type results. No genuine generalization for arbitrary $\oplus$ with $e > 0$ is provided.

The following results clarify structural restrictions inherent in the framework proposed by Adewale et al.

Theorem 1.3. The contraction $b(T\xi, T\flat) \leq k \cdot b(\xi, \flat)$ with $0 \leq k < 1$ is valid only when $e = 0$.

Proof. If $e > 0$, take $\xi = \flat$. Then $b(\xi, \flat) = e$, and the inequality gives:

$$b(T\xi, T\xi) \leq k \cdot b(\xi, \xi) = k \cdot e.$$

But $b(T\xi, T\xi) = e$ by (OM2). Hence $e \leq k \cdot e$, which implies $k \geq 1$, contradicting $k < 1$. Therefore $e = 0$. $\square$

Theorem 1.4. *For any operation $\oplus$ with identity e , the only element $k \in [e, \infty)$ satisfying $k \oplus a \leq a$ for all $a \geq e$ is $k = e$.*

Proof. Taking $a = e$, we have $k \oplus e \leq e$. Since e is the identity, $k \oplus e = k$. Hence $k \leq e$. But $k \in [e, \infty)$, so $k \geq e$. Therefore $k = e$. $\square$

Theorem 1.4 shows that the natural contraction $k \oplus b(\xi, \flat)$ cannot produce a Banach-type contraction. Based on the above ideas, we define a bi-operational metric space where the triangular inequality is controlled by one binary operation while another binary operation controls the contraction process. Moreover, we introduce a genuinely new framework using *two* binary operations:

1. $\oplus$ for the triangle inequality (with identity e),
2. $\odot$ for the contraction (not necessarily requiring an identity).

We introduce the concept of a $\odot$ -contractive element and prove a Banach-type fixed point theorem that does not reduce to the classical case. We provide an explicit example demonstrating the power of our approach.

2 Bi-Operational Metric Spaces

We now introduce our new framework.

Definition 2.1. *Let $\mathcal{O}$ be a nonempty set. Let $\oplus$ be a binary operation on $[e, \infty)$ with two-sided identity e . Let $\odot$ be another binary operation on $[e, \infty)$. A mapping $\mathcal{B} : \mathcal{O} \times \mathcal{O} \rightarrow [e, \infty)$ is a bi-operational metric if, for all $\xi, \flat, z \in \mathcal{O}$,*

- (B1) $\mathcal{B}(\xi, \flat) \geq e$;
- (B2) $\mathcal{B}(\xi, \flat) = e \iff \xi = \flat$;
- (B3) $\mathcal{B}(\xi, \flat) = \mathcal{B}(\flat, \xi)$;
- (B4) $\mathcal{B}(\xi, \flat) \leq \mathcal{B}(\xi, z) \oplus \mathcal{B}(z, \flat)$.

In this case the quadruple $(\mathcal{O}, \mathcal{B}, \oplus, \odot)$ is called a bi-operational metric space.

Definition 2.2. A sequence $\{\xi_n\}$ in $\mathcal{O}$ is bi-operational convergent to $\xi \in \mathcal{O}$ if $\mathcal{B}(\xi_n, \xi) \rightarrow e$ as $n \rightarrow \infty$.

Definition 2.3. A sequence $\{\xi_n\}$ is called bi-operational Cauchy if $\mathcal{B}(\xi_n, \xi_m) \rightarrow e$ as $m, n \rightarrow \infty$.

Definition 2.4. A bi-operational metric space is complete if every bi-operational Cauchy sequence converges.

Definition 2.5 (Monotone operation). Let $\star$ be a binary operation on $[e, \infty)$. We say that $\star$ is monotone if, for all $a, g, h, d \in [e, \infty)$,

$$a \leq g \text{ and } h \leq d \implies a \star h \leq g \star d.$$

Definition 2.6 (Continuous operation). Let $\star$ be a binary operation on $[e, \infty)$. We say that $\star$ is continuous if, whenever $h_n \rightarrow h$ and $g_n \rightarrow g$ in $[e, \infty)$,

$$h_n \star g_n \longrightarrow h \star g.$$

Definition 2.7 (Associative operation). A binary operation $\star$ on $[e, \infty)$ is associative if $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in [e, \infty)$.

Remark 2.8. Definitions 2.5–2.7 are stated for a generic binary operation $\star$ on $[e, \infty)$ and therefore apply verbatim to $\oplus$ and to $\odot$, without any further comment. In particular, Hypothesis 1 of Theorem 4.1, “ $\oplus, \odot$ are continuous, associative and monotone”, means that both operations satisfy Definitions 2.5, 2.6 and 2.7 with $\star = \oplus$ and with $\star = \odot$, respectively.

Remark 2.9. The key innovation is that the contraction operation $\odot$ does not require an identity. This allows us to define operations that genuinely shrink distances without the constraints imposed by the identity property.

Lemma 2.10. Let $(\mathcal{O}, \mathcal{B})$ be a bi-operational metric space whose operation $\oplus$ is continuous. Then every bi-operational convergent sequence $\{\xi_n\}$ in $\mathcal{O}$ converges to only one point.

Proof. Suppose that $\xi_n \rightarrow \xi$ and $\xi_n \rightarrow \flat$. Then by (B4), we get

$$\mathcal{B}(\xi, \flat) \leq \mathcal{B}(\xi, \xi_n) \oplus \mathcal{B}(\xi_n, \flat).$$

Since $\oplus$ is continuous, we get

$$\mathcal{B}(\xi, \xi_n) \oplus \mathcal{B}(\xi_n, \flat) \rightarrow e \oplus e = e.$$

Hence $\mathcal{B}(\xi, \flat) \leq e$, and since $\mathcal{B}(\xi, \flat) \geq e$ by (B1), we obtain $\mathcal{B}(\xi, \flat) = e$, and so, $\xi = \flat$. $\square$

2.1 Contractive Elements

Definition 2.11. Assume that $\odot$ is an associative binary operation on $[e, \infty)$. An element $k \in [e, \infty)$ is called $\odot$ -contractive if

- (C1) $k \odot a < a$ for all $a > e$;
- (C2) $k^{\odot n} \odot a \rightarrow e$ as $n \rightarrow \infty$ for all $a \geq e$,

where

$$k^{\odot n} = \underbrace{k \odot k \odot \cdots \odot k}_{n \text{ times}}.$$

Remark 2.12. The condition $k \odot a < a$ ensures that each application of k shrinks distances. The condition $k^{\odot n} \odot a \rightarrow e$ ensures that repeated application converges to the identity of the triangle operation.

Remark 2.13 (Well-definedness of $k^{\odot n}$). The associativity of $\odot$ is not an incidental assumption made later in the development, it is a standing hypothesis of Definition 2.11, stated in its first line, and it is exactly what makes the symbol $k^{\odot n}$ meaningful. Indeed, for a non-associative operation the n -fold combination $k \odot k \odot \cdots \odot k$ depends on the bracketing, so no single element $k^{\odot n}$ would be determined. Under Definition 2.7 all bracketings coincide, and $k^{\odot n}$ is unambiguously defined by the recursion

$$k^{\odot 1} = k, \quad k^{\odot(n+1)} = k \odot k^{\odot n} \quad (n \geq 1),$$

which is the form used throughout the proofs below. Consequently condition (C2) and the Operational Convergence Property (Definition 2.18) are well posed. For the specific operation of Example 4.3, associativity is verified directly in Appendix A, part (A.11), and the closed form of $k^{\odot n}$ is computed in part (A.8); no circularity is involved, since the appendix merely confirms for that example a property which is assumed abstractly here.

Lemma 2.14 ($\oplus$ is increasing). Let $\oplus$ be a binary operation on $[e, \infty)$ with two-sided identity e (i.e. $a \oplus e = e \oplus a = a$ for all $a \geq e$) which is monotone in the sense of Definition 2.5. Then for all $a, b \in [e, \infty)$,

$$a \oplus b \geq a \quad \text{and} \quad a \oplus b \geq b.$$

Proof. Since $b \geq e$ and $a \leq a$, monotonicity gives $a \oplus b \geq a \oplus e = a$. Since $a \geq e$ and $b \leq b$, monotonicity gives $a \oplus b \geq e \oplus b = b$. $\square$

Definition 2.15 (Infinite $\oplus$ -combination). Assume throughout this definition that $\oplus$ is associative and monotone. Let $(x_i)_{i \geq m}$ be a sequence in $[e, \infty)$. For

$N \geq m$ define the finite partial combination

$$S_N := x_m \oplus x_{m+1} \oplus \cdots \oplus x_N,$$

which is well defined because $\oplus$ is associative. By Lemma 2.14,

$$S_{N+1} = S_N \oplus x_{N+1} \geq S_N,$$

so the sequence $(S_N)_{N \geq m}$ is non-decreasing in the usual order of $[e, \infty)$. We therefore define

$$\bigoplus_{i=m}^{\infty} x_i := \lim_{N \rightarrow \infty} S_N \in [e, +\infty],$$

allowing the value $+\infty$ when $(S_N)_N$ is unbounded, and we say that the infinite combination converges when this limit is finite.

Corollary 2.16. *If $\bigoplus_{i=m}^{\infty} x_i$ converges, then for every $n > m$,*

$$\bigoplus_{i=m}^{n-1} x_i \leq \bigoplus_{i=m}^{\infty} x_i.$$

Proof. Immediate from Definition 2.15: the sequence $(S_N)_{N \geq m}$ of partial combinations is non-decreasing, so every partial value S_{n-1} satisfies $S_{n-1} \leq \lim_{N \rightarrow \infty} S_N$. $\square$

Remark 2.17. *Corollary 2.16 is precisely the statement that was invoked informally in the proof of Theorem 4.1 under the phrase “since the triangle-inequality operation is increasing”. The proof below now refers to it explicitly.*

Definition 2.18 (Operational Convergence Property (OCP)). *We say that $(\oplus, \odot)$ satisfies the Operational Convergence Property (OCP) if for every $a \geq e$ and every $\odot$ -contractive element k , the infinite combination $\bigoplus_{i=m}^{\infty} (k^{\odot i} \odot a)$ (Definition 2.15) converges for every $m \geq 1$, and*

$$\bigoplus_{i=m}^{\infty} (k^{\odot i} \odot a) \longrightarrow e \quad \text{as } m \rightarrow \infty.$$

3 Examples

In this section we show that the bi-operational metric framework recovers several well-known distance structures as special cases.

3.1 Classical Metric Spaces

Let $\mathcal{O}$ be a nonempty set. The usual metric setting is obtained by considering a mapping

$$d : \mathcal{O} \times \mathcal{O} \rightarrow [0, +\infty)$$

satisfying the following conditions:

- (d1) $d(\xi, \flat) = 0$ if and only if $\xi = \flat$;
- (d2) $d(\xi, \flat) = d(\flat, \xi)$;
- (d3) $d(\xi, \flat) \leq d(\xi, z) + d(z, \flat)$ for all $\xi, \flat, z \in \mathcal{O}$.

Proposition 3.1. *Every standard metric space $(\mathcal{O}, d)$ is a bi-operational metric space $(\mathcal{O}, d, \oplus)$ under the choice $a \oplus b = a + b$ with identity $e = 0$.*

Proof. Conditions (B1)–(B3) follow immediately from (d1)–(d2) and non-negativity of d . Condition (B4) is exactly (d3), since $d(\xi, \flat) \leq d(\xi, z) + d(z, \flat) = d(\xi, z) \oplus d(z, \flat)$. The operation $\oplus = +$ is continuous, associative, and monotone on $[0, \infty)$ with identity $e = 0$. $\square$

3.2 Multiplicative Metric Spaces

A multiplicative metric on a nonempty set $\mathcal{O}$ is a mapping $d : \mathcal{O} \times \mathcal{O} \rightarrow [1, +\infty)$ satisfying:

- (m1) $d(\xi, \flat) = 1 \iff \xi = \flat$;
- (m2) $d(\xi, \flat) = d(\flat, \xi)$;
- (m3) $d(\xi, \flat) \leq d(\xi, z) \cdot d(z, \flat)$ for all $\xi, \flat, z \in \mathcal{O}$.

Proposition 3.2. *Every multiplicative metric space $(\mathcal{O}, d)$ is a bi-operational metric space $(\mathcal{O}, d, \oplus)$ under the choice $a \oplus b = a \cdot b$ with identity $e = 1$.*

Proof. Conditions (B1)–(B3) follow immediately from (m1)–(m2). Condition (B4) is exactly (m3), since $d(\xi, \flat) \leq d(\xi, z) \cdot d(z, \flat) = d(\xi, z) \oplus d(z, \flat)$. The operation $\oplus = \cdot$ is continuous, associative, and monotone on $[1, \infty)$ with identity $e = 1$. $\square$

3.3 Ultrametric Spaces

An ultrametric on $\mathcal{O}$ is a mapping $d : \mathcal{O} \times \mathcal{O} \rightarrow [0, +\infty)$ satisfying (d1)–(d2) above and

- (d3) $d(\xi, \flat) \leq \max\{d(\xi, z), d(z, \flat)\}$ for all $\xi, \flat, z \in \mathcal{O}$.

Proposition 3.3. *Every ultrametric space $(\mathcal{O}, d)$ is a bi-operational metric space $(\mathcal{O}, d, \oplus)$ under the choice $a \oplus b = \max\{a, b\}$ with identity $e = 0$.*

Proof. Conditions (B1)–(B3) follow immediately from (d1)–(d2). Condition (B4) is exactly (u3), since $d(\xi, \flat) \leq \max\{d(\xi, z), d(z, \flat)\} = d(\xi, z) \oplus d(z, \flat)$. The operation $\oplus = \max$ is continuous, associative, and monotone on $[0, \infty)$ with identity $e = 0$. $\square$

Remark 3.4. *Propositions 3.1–3.3 concern only the triangle-inequality operation $\oplus$, because Definition 2.1 requires only $\oplus$ to define a bi-operational metric space. The contraction operation $\odot$ is independent of the metric structure and enters only when a specific contraction mapping T is studied. This is precisely the separation that distinguishes the bi-operational framework from the operational metric spaces of Adewale et al., where a single operation governs both the triangle inequality and the contraction. The choice of $\odot$ is made freely and independently in each fixed point problem, as illustrated in the consequence results.*

3.4 A Comparative Summary

Table 1 collects the special cases discussed above together with the two frameworks they are being compared against. Reading the table by columns makes the contribution of the present paper visible at a glance: in every row except the last one the contraction mechanism is either dictated by the triangle-inequality operation or imported from outside the structure as ordinary real multiplication, whereas in the bi-operational setting the two operations are chosen independently of one another.

Remark 3.5. *The fourth row of Table 1 deserves a comment, since b -metric spaces are the most frequently cited generalization of the triangle inequality. Writing the b -triangle inequality as $d(\xi, \flat) \leq s[d(\xi, z) + d(z, \flat)]$ with $s > 1$ suggests the choice $a \oplus_s b := s(a + b)$. This operation is continuous and monotone on $[0, \infty)$, but it is neither associative nor unital: one has $(a \oplus_s b) \oplus_s c = s^2a + s^2b + sc$ while $a \oplus_s (b \oplus_s c) = sa + s^2b + s^2c$, and $a \oplus_s 0 = sa \neq a$ for every $a > 0$. Hence $\oplus_s$ does not meet the requirement of Definition 2.1, and b -metric spaces are not subsumed by the framework of the present paper. Extending the bi-operational setting to triangle-inequality operations that are only approximately associative and possess only an approximate identity, so as to cover the b -metric case, is an interesting problem that we leave for future work.*

Table 1 Comparison of the bi-operational framework with the classical distance structures and with the operational metric spaces of Adewale et al. The decisive difference is recorded in the fourth column: only in the last row is the contraction operation a free parameter of the theory.

Framework	Triangle Operation $\oplus$	Identity e	Contraction Operation $\odot$	Contraction Condition
Metric	$a \oplus b = a + b$	0	$a \odot b = a \cdot b$	$d(T\xi, T\mathfrak{b}) \leq k d(\xi, \mathfrak{b})$, $k \in [0, 1)$
Multiplicative [10]	$a \oplus b = a \cdot b$	1	$a \odot b = b^{\ln a}$	$d(T\xi, T\mathfrak{b}) \leq d(\xi, \mathfrak{b})^{\ln k}$
Ultrametric [16]	$a \oplus b = \max\{a, b\}$	0	$a \odot b = a \cdot b$	$d(T\xi, T\mathfrak{b}) \leq k d(\xi, \mathfrak{b})$, $k \in [0, 1)$
b -metric [2, 11]	$a \oplus b = s(a + b)$, $s > 1$	none	—	not covered; see Remark 3.5
Operational [1]	arbitrary, with identity	e	none: real scalar multiplication is used	$b(T\xi, T\mathfrak{b}) \leq k \cdot b(\xi, \mathfrak{b})$, forces $e = 0$
Bi-operational (this paper)	arbitrary, with identity	e	arbitrary, independent of $\oplus$, identity not required	$\mathcal{B}(T\xi, T\mathfrak{b}) \leq k \odot \mathcal{B}(\xi, \mathfrak{b})$, k $\odot$ -contractive

4 Banach Fixed Point Theorem in Bi-operational Metric Spaces

Theorem 4.1. Let $(\mathcal{O}, \mathcal{B}, \oplus, \odot)$ be a complete bi-operational metric space. Assume

1. $\oplus, \odot$ are continuous, associative and monotone,
2. there exists a $\odot$ -contractive element $k \in [e, \infty)$, where e is the identity element of $\oplus$ and a two-sided absorbing element of $\odot$, i.e., $e \odot g = g \odot e = e$ for all $g \geq e$,
3. $(\oplus, \odot)$ satisfies OCP.

If $T : \mathcal{O} \rightarrow \mathcal{O}$ satisfies

$$\mathcal{B}(T\xi, T\mathfrak{b}) \leq k \odot \mathcal{B}(\xi, \mathfrak{b}),$$

then T has a unique fixed point.

Proof. Let $\xi_{n+1} = T\xi_n$. If there is $\xi_q = \xi_{q+1}$, then ξ_q is a fixed point for T . Thus, we may suppose that for all n , $\xi_{n+1} \neq \xi_n$.

Step 1: the basic estimate. We claim that, with $b_0 := \mathcal{B}(\xi_1, \xi_0)$,

$$\mathcal{B}(\xi_{n+1}, \xi_n) \leq k^{\odot n} \odot b_0 \quad \text{for every } n \geq 1. \quad (1)$$

We argue by induction on n . For $n = 1$, the contraction hypothesis gives

$$\mathcal{B}(\xi_2, \xi_1) = \mathcal{B}(T\xi_1, T\xi_0) \leq k \odot \mathcal{B}(\xi_1, \xi_0) = k^{\odot 1} \odot b_0,$$

which is (1) for $n = 1$. Assume now that (1) holds for some $n \geq 1$. Applying the contraction hypothesis to the pair (ξ_{n+1}, ξ_n) and then the induction hypothesis, we obtain

$$\mathcal{B}(\xi_{n+2}, \xi_{n+1}) = \mathcal{B}(T\xi_{n+1}, T\xi_n) \leq k \odot \mathcal{B}(\xi_{n+1}, \xi_n) \leq k \odot (k^{\odot n} \odot b_0),$$

where the second inequality uses the monotonicity of $\odot$ in its second argument (Definition 2.5). Since $\odot$ is associative (Definition 2.7),

$$k \odot (k^{\odot n} \odot b_0) = (k \odot k^{\odot n}) \odot b_0 = k^{\odot(n+1)} \odot b_0,$$

by the recursion recorded in Remark 2.13. This is (1) for $n + 1$, and the induction is complete.

Step 2: the Cauchy property. For $n > m \geq 1$, we first use the symmetry (B3) and then apply the triangle inequality (B4) repeatedly, splitting off the leftmost term at each stage, so that the resulting combination is taken in increasing order of the index; with the estimate (1) and the monotonicity of $\oplus$ this gives

$$\begin{aligned} \mathcal{B}(\xi_n, \xi_m) &= \mathcal{B}(\xi_m, \xi_n) \\ &\leq \mathcal{B}(\xi_m, \xi_{m+1}) \oplus \mathcal{B}(\xi_{m+1}, \xi_{m+2}) \oplus \cdots \oplus \mathcal{B}(\xi_{n-1}, \xi_n) \\ &\leq \bigoplus_{i=m}^{n-1} (k^{\odot i} \odot b_0) \quad \text{where } b_0 = \mathcal{B}(\xi_1, \xi_0) \\ &\leq \bigoplus_{i=m}^{\infty} (k^{\odot i} \odot b_0) \quad (\text{by Corollary 2.16}) \end{aligned}$$

By OCP, we have $\bigoplus_{i=m}^{\infty} (k^{\odot i} \odot b_0) \rightarrow e$ as $m \rightarrow \infty$. Hence $\{\xi_n\}$ is Cauchy and, by completeness, converges to some $u \in \mathcal{O}$.

Step 3: u is a fixed point. Observe that, for every n ,

$$\mathcal{B}(Tu, \xi_{n+1}) = \mathcal{B}(Tu, T\xi_n) \leq k \odot \mathcal{B}(u, \xi_n).$$

Since $\xi_n \rightarrow u$ we have $\mathcal{B}(u, \xi_n) \rightarrow e$, and the continuity of $\odot$ (Definition 2.6) gives

$$k \odot \mathcal{B}(u, \xi_n) \longrightarrow k \odot e = e,$$

the last equality being exactly the *two-sided* absorption assumed in Hypothesis 2. On the other hand $\mathcal{B}(Tu, \xi_{n+1}) \geq e$ for every n by (B1). Combining the two facts,

$$e \leq \mathcal{B}(Tu, \xi_{n+1}) \leq k \odot \mathcal{B}(u, \xi_n) \longrightarrow e,$$

so the squeeze principle yields $\mathcal{B}(Tu, \xi_{n+1}) \rightarrow e$; that is, $\xi_{n+1} \rightarrow Tu$. Since $\xi_{n+1} \rightarrow u$ as well, Lemma 2.10 (uniqueness of bi-operational limits) forces $Tu = u$.

Step 4: uniqueness. If u, v are fixed points, then

$$\mathcal{B}(u, v) = \mathcal{B}(Tu, Tv) \leq k \odot \mathcal{B}(u, v).$$

If $\mathcal{B}(u, v) > e$, then by the contractivity of k (condition (C1)), we get

$$k \odot \mathcal{B}(u, v) < \mathcal{B}(u, v),$$

a contradiction. So, $\mathcal{B}(u, v) = e$, hence $u = v$. $\square$

Remark 4.2. Hypothesis 2 requires two-sided absorption, $e \odot g = g \odot e = e$ for all $g \geq e$. The left-hand identity $e \odot g = e$ is what makes e an absorbing element in the algebraic sense; the right-hand identity $g \odot e = e$ is the one actually used in Step 3, where the limit $k \odot \mathcal{B}(u, \xi_n) \rightarrow k \odot e$ must be identified with e . Both identities are verified in each of the Corollaries 5.1, 5.2 and 5.3, and for the operation of Example 4.3 in Appendix A, part (A.7); stating the hypothesis in two-sided form therefore brings the theorem into exact agreement with what is used and verified in the rest of the paper.

The following example verifies the assumptions imposed in our main theorem and illustrates the obtained results.

Example 4.3. Let $\mathcal{O} = [2, 3]$ and let $\oplus$ and $\odot$ be the binary operations defined on $[2, \infty)$ by

$$a \oplus b = (a - 1)(b - 1) + 1, \quad a \odot b = (a - 1)^{\ln(b-1)} + 1.$$

Consider the function $\mathcal{B} : \mathcal{O} \times \mathcal{O} \rightarrow [2, \infty)$ defined by

$$\mathcal{B}(\xi, \flat) = |\xi - \flat| + 2.$$

Define $T : \mathcal{O} \rightarrow \mathcal{O}$ by

$$T(\xi) = 2 + \frac{\xi - 2}{10}.$$

Choose $k = 2.5$.

The identity element of $\oplus$ is $e = 2$, and $(\mathcal{O}, \mathcal{B}, \oplus, \odot)$ is a complete bi-operational metric space on which Hypotheses 1–3 of Theorem 4.1 are satisfied; all of these verifications, including the fact that T maps $\mathcal{O}$ into itself, that $k = 2.5$ is $\odot$ -contractive, and that $(\oplus, \odot)$ satisfies OCP, are carried out in detail in Appendix A. Note that $\mathcal{O} = [2, 3]$ is a closed subinterval of $[2, \infty)$, which is the set on which the two operations and the bi-operational metric take their values; the choice of the compact interval $[2, 3]$ for the underlying space is what makes the elementary estimate below possible.

It remains to verify that the mapping T satisfies the contractive hypothesis of Theorem 4.1. For arbitrary $\xi, \flat \in \mathcal{O}$, let

$$t = |\xi - \flat| \in [0, 1].$$

Then

$$\mathcal{B}(T\xi, T\flat) = \left| 2 + \frac{\xi - 2}{10} - \left(2 + \frac{\flat - 2}{10} \right) \right| + 2 = \frac{t}{10} + 2.$$

On the other hand,

$$k \odot \mathcal{B}(\xi, \flat) = (2.5 - 1)^{\ln(|\xi - \flat| + 1)} + 1 = (1.5)^{\ln(t+1)} + 1.$$

Since $x^{\ln y} = y^{\ln x}$ for all $x, y > 0$,

$$(1.5)^{\ln(t+1)} = (t+1)^{\ln(1.5)}.$$

We now show that

$$(t+1)^\alpha \geq 1 + \frac{t}{10} \quad \text{for all } t \in [0, 1], \quad \text{where } \alpha = \ln(1.5) \approx 0.4055. \quad (2)$$

Since $\alpha \in (0, 1)$, the function $s \mapsto s^\alpha$ is concave, so Bernoulli's inequality applies in the form $(1+t)^\alpha \leq 1 + \alpha t$, that is, in the direction opposite to the one that would be needed here. We therefore establish (2) directly, by a monotonicity argument. Put

$$f(t) = (t+1)^\alpha - 1 - \frac{t}{10}, \quad t \in [0, 1].$$

Then $f(0) = 1 - 1 - 0 = 0$, and f is differentiable on $[0, 1]$ with

$$f'(t) = \alpha(t+1)^{\alpha-1} - \frac{1}{10}.$$

Because $\alpha - 1 < 0$, the map $t \mapsto (t+1)^{\alpha-1}$ is decreasing on $[0, 1]$, and hence attains its minimum at $t = 1$;

$$(t+1)^{\alpha-1} \geq 2^{\alpha-1} = 2^{-0.5945} \approx 0.6623 \quad (0 \leq t \leq 1).$$

Consequently

$$f'(t) \geq \alpha \cdot 2^{\alpha-1} - \frac{1}{10} \approx 0.4055 \times 0.6623 - 0.1 \approx 0.1685 > 0 \quad (0 \leq t \leq 1),$$

so f is strictly increasing on $[0, 1]$. Together with $f(0) = 0$ this yields $f(t) \geq 0$ for every $t \in [0, 1]$, which is exactly (2).

Therefore,

$$k \odot \mathcal{B}(\xi, \flat) = (1.5)^{\ln(t+1)} + 1 = (t+1)^{\ln(1.5)} + 1 \geq 2 + \frac{t}{10} = \mathcal{B}(T\xi, T\flat).$$

Hence

$$\mathcal{B}(T\xi, T\flat) \leq k \odot \mathcal{B}(\xi, \flat),$$

for all $\xi, \flat \in \mathcal{O}$. Finally, if $\xi = T(\xi)$, then

$$\xi = 2 + \frac{\xi - 2}{10},$$

which implies

$$10\xi = 20 + \xi - 2,$$

and consequently

$$9\xi = 18.$$

Thus $\xi = 2$. Consequently, the mapping T admits a unique fixed point, which is given by $\xi = 2$. Hence, all the hypotheses required by Theorem 4.1 are fulfilled.

Remark 4.4. The estimate (2) is sharp at the left endpoint, since $f(0) = 0$; this reflects the fact that $\mathcal{B}(T\xi, T\flat) = k \odot \mathcal{B}(\xi, \flat) = 2 = e$ when $\xi = \flat$. It is also worth noticing that the inequality $(t+1)^\alpha \geq 1 + t/10$ fails for large t : the function f above has $f'(t) = 0$ at $t = (10\alpha)^{1/(1-\alpha)} - 1 \approx 9.53$, and f becomes negative for t beyond roughly $t \approx 30.5$. The restriction of the underlying space to the bounded interval $\mathcal{O} = [2, 3]$, which confines $t = |\xi - \flat|$ to $[0, 1]$, is therefore essential and not merely a convenience.

5 Consequences of the Main Theorem

In this section we show that Theorem 4.1 recovers the classical fixed point results corresponding to each special case above, by specifying the contraction operation $\odot$ in each setting.

5.1 Banach Contraction Principle

Corollary 5.1. *Let $(\mathcal{O}, d)$ be a complete standard metric space and $T : \mathcal{O} \rightarrow \mathcal{O}$ a mapping. Suppose there exists $k \in [0, 1)$ such that*

$$d(T\xi, T\flat) \leq k d(\xi, \flat) \quad \text{for all } \xi, \flat \in \mathcal{O}.$$

Then T has a unique fixed point.

Proof. Set $a \oplus b = a + b$ and $a \odot b = a \cdot b$ with $e = 0$. By Proposition 3.1, $(\mathcal{O}, d, \oplus)$ is a complete bi-operational metric space. The operation $\odot$ is continuous, associative and monotone. Two-sided absorption holds since $0 \odot a = a \odot 0 = 0 = e$. For $k \in [0, 1)$:

(C1) $k \odot a = ka < a$ for all $a > 0$.

(C2) $k^{\odot n} \odot a = k^n a \rightarrow 0 = e$.

OCP holds since $\bigoplus_{i=m}^{\infty} (k^i a) = \sum_{i=m}^{\infty} k^i a = k^m a / (1 - k) \rightarrow 0 = e$. The contraction condition $d(T\xi, T\flat) \leq k \odot d(\xi, \flat) = k d(\xi, \flat)$ coincides with the hypothesis. The result follows from Theorem 4.1. $\square$

5.2 Multiplicative Contraction Principle

Corollary 5.2. *Let $(\mathcal{O}, d)$ be a complete multiplicative metric space and $T : \mathcal{O} \rightarrow \mathcal{O}$ a mapping. Suppose there exists $k \in [1, \exp(1))$, where $\exp(1) \approx 2.71828$ denotes Euler's number, such that*

$$d(T\xi, T\flat) \leq d(\xi, \flat)^{\ln k} \quad \text{for all } \xi, \flat \in \mathcal{O}.$$

Then T has a unique fixed point.

Proof. Set $a \oplus b = a \cdot b$ and $a \odot b = b^{\ln a}$ with $e = 1$. By Proposition 3.2, $(\mathcal{O}, d, \oplus)$ is a complete bi-operational metric space. The operation $\odot$ is continuous, associative and monotone. Two-sided absorption holds since $1 \odot a = a^{\ln 1} = a^0 = 1 = e$ and $a \odot 1 = 1^{\ln a} = 1 = e$. For $k \in [1, \exp(1))$, so that $\ln k \in [0, 1)$:

(C1) $k \odot a = a^{\ln k} < a$ for all $a > 1$, since $\ln k < 1$.

(C2) $k^{\odot n} \odot a = a^{(\ln k)^n} \rightarrow a^0 = 1 = e$ since $(\ln k)^n \rightarrow 0$.

OCP holds since $\bigoplus_{i=m}^{\infty} (k^{\odot i} \odot a) = \prod_{i=m}^{\infty} a^{(\ln k)^i} = a^{(\ln k)^m / (1 - \ln k)} \rightarrow 1 = e$. The contraction condition $d(T\xi, T\flat) \leq k \odot d(\xi, \flat) = d(\xi, \flat)^{\ln k}$ coincides with the hypothesis. The result follows from Theorem 4.1. $\square$

5.3 Ultrametric Contraction Principle

Corollary 5.3. *Let $(\mathcal{O}, d)$ be a complete ultrametric space and $T : \mathcal{O} \rightarrow \mathcal{O}$ a mapping. Suppose there exists $k \in [0, 1)$ such that*

$$d(T\xi, T\flat) \leq k d(\xi, \flat) \quad \text{for all } \xi, \flat \in \mathcal{O}.$$

Then T has a unique fixed point.

Proof. Set $a \oplus b = \max\{a, b\}$ and $a \odot b = a \cdot b$ with $e = 0$. By Proposition 3.3, $(\mathcal{O}, d, \oplus)$ is a complete bi-operational metric space. The operation $\odot$ is continuous, associative and monotone. Two-sided absorption holds since $0 \odot a = a \odot 0 = 0 = e$. For $k \in [0, 1)$:

(C1) $k \odot a = ka < a$ for all $a > 0$.

(C2) $k^{\odot n} \odot a = k^n a \rightarrow 0 = e$.

OCP holds since $\bigoplus_{i=m}^{\infty} (k^{\odot i} a) = \sup_{i \geq m} (k^i a) = k^m a \rightarrow 0 = e$. The contraction condition $d(T\xi, T\flat) \leq k \odot d(\xi, \flat) = k d(\xi, \flat)$ coincides with the hypothesis. The result follows from Theorem 4.1. $\square$

6 Conclusion

In this paper, we have identified the fundamental limitations of the approach of Adewale et al., in which the contraction of the form $k \cdot b(\xi, \flat)$ is only meaningful when $e = 0$, and in which real scalar multiplication is mixed with the underlying operational metric structure. To overcome these issues, we introduced a genuinely new framework based on two binary operations: $\oplus$, which governs the triangle-type inequality and is equipped with an identity element, and $\odot$, which governs the contraction mechanism and does not require an identity. Within this setting, we introduced the notion of a $\odot$ -contractive element k , characterized by the property that $k \odot a < a$ for all $a > e$, together with the asymptotic condition $k^{\odot n} \odot a \rightarrow e$. Under these assumptions, we established a Banach-type fixed point theorem in the bi-operational framework. Furthermore, we provided illustrative examples, including the classical case and a non-trivial construction on $[2, 3]$ defined

by $a \oplus b = (a - 1)(b - 1) + 1$ and $a \odot b = (a - 1)^{\ln(b-1)} + 1$. These results demonstrate that the proposed framework constitutes a genuine and non-trivial generalization of Banach's contraction principle, in which the contraction mechanism is a free parameter of the theory rather than a consequence of the triangle-inequality operation.

The framework developed here sits at the meeting point of metric fixed point theory and algebra, and we believe that this is where its principal interest lies. A bi-operational metric space is, in effect, a set equipped with a distance taking values in an ordered algebraic structure carrying two operations, one of which need not have an identity; structures of this kind are familiar from the theory of ordered semigroups, semirings and residuated lattices, and it would be natural to ask which fixed point statements survive when $[e, \infty)$ is replaced by an abstract ordered monoid or a complete lattice. In a different direction, the axioms (C1)–(C2) and the Operational Convergence Property are conditions on the asymptotics of the iterates $k^{\odot n}$, which invites a dynamical reading of the theory. We therefore expect that future work may profitably explore the applications and the connections of bi-operational metric spaces with other branches of mathematics, among them ordered algebraic structures and lattice theory, topological algebra, and the theory of integral and fractional differential equations, where contraction principles are the standard route to existence and uniqueness results [7, 9]. Finally, as observed in Remark 3.5, the b -metric case is not covered by the present axioms, and finding the right relaxation of the identity requirement so as to include it remains an open problem.

Conflict of Interest

The author declares no conflict of interest.

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Appendix

A Detailed Verification of Example 4.3

Example A.1 (Full Verification of Algebraic Structure).

Let $\mathcal{O} = [2, 3]$, and let the operations

$$a \oplus b = (a - 1)(b - 1) + 1, \quad a \odot b = (a - 1)^{\ln(b-1)} + 1$$

be defined on $[2, \infty)$, which is the codomain of the bi-operational metric $\mathcal{B}(\xi, \flat) = |\xi - \flat| + 2$ of Example 4.3.

Throughout this appendix, $\mathcal{O} = [2, 3]$ denotes the underlying space of Example 4.3, while $[2, \infty)$ denotes the set on which $\oplus$, $\odot$ and $\mathcal{B}$ take their values. The two intervals play different roles and are no longer used interchangeably: parts (A.1)–(A.5), (A.7), (A.8), (A.9) and (A.11) below are statements about the operations, and are therefore proved on all of $[2, \infty)$; parts (A.6) and (A.10) are statements about the space and the mapping T , and are proved on $\mathcal{O} = [2, 3]$.

(A.1) Identity of $\oplus$. For all $a \geq 2$,

$$a \oplus 2 = (a - 1)(1) + 1 = a, \quad 2 \oplus a = (1)(a - 1) + 1 = a.$$

Hence $e = 2$ is the identity element of $\oplus$.

(A.2) Well-definedness. Since $a - 1 \geq 1$ and $b - 1 \geq 1$ for all $a, b \geq 2$, both operations are well-defined and map $[2, \infty)$ into itself.

(A.3) Monotonicity of $\oplus$. If $a \leq c$ and $b \leq d$, then

$$(a - 1)(b - 1) \leq (c - 1)(d - 1),$$

so $a \oplus b \leq c \oplus d$.

(A.4) Monotonicity of $\odot$. Fix $b \geq 2$. Since $\ln(b - 1) \geq 0$, the map $a \mapsto (a - 1)^{\ln(b-1)} + 1$ is non-decreasing in a for $a \geq 2$. Fix $a \geq 2$. Since $a - 1 \geq 1$ and $\ln(\cdot)$ is increasing, the map $b \mapsto (a - 1)^{\ln(b-1)} + 1$ is non-decreasing in b . Hence $\odot$ is monotone in each argument.

(A.5) Continuity. Both operations are compositions of continuous functions on $[2, \infty)$, hence continuous.

(A.6) $\mathcal{B}$ is a complete bi-operational metric on $\mathcal{O} = [2, 3]$. Conditions (B1)–(B3) of Definition 2.1 are immediate: $\mathcal{B}(\xi, \flat) = |\xi - \flat| + 2 \geq 2 = e$; equality holds if and only if $|\xi - \flat| = 0$, that is $\xi = \flat$; and $\mathcal{B}$ is symmetric because $|\cdot|$ is. For (B4), let $\xi, \flat, z \in \mathcal{O}$ and put $u = |\xi - z|$, $v = |z - \flat|$. Then

$$\begin{aligned} \mathcal{B}(\xi, z) \oplus \mathcal{B}(z, \flat) &= (u + 2 - 1)(v + 2 - 1) + 1 \\ &= (u + 1)(v + 1) + 1 = uv + u + v + 2, \end{aligned}$$

whereas, by the ordinary triangle inequality on $\mathbb{R}$,

$$\mathcal{B}(\xi, \flat) = |\xi - \flat| + 2 \leq u + v + 2.$$

Since $uv \geq 0$, we conclude $\mathcal{B}(\xi, \flat) \leq uv + u + v + 2 = \mathcal{B}(\xi, z) \oplus \mathcal{B}(z, \flat)$, which is (B4). As for completeness, $\mathcal{B}(\xi_n, \xi_m) \rightarrow e = 2$ is equivalent to $|\xi_n - \xi_m| \rightarrow 0$, so a bi-operational Cauchy sequence in $\mathcal{O}$ is exactly a Cauchy sequence of real numbers lying in the closed bounded interval $[2, 3]$. Such a sequence converges in $[2, 3]$ to some ξ , and then $\mathcal{B}(\xi_n, \xi) = |\xi_n - \xi| + 2 \rightarrow 2 = e$, i.e. $\xi_n \rightarrow \xi$ in the bi-operational sense. Hence $(\mathcal{O}, \mathcal{B}, \oplus, \odot)$ is complete. Finally, T maps $\mathcal{O}$ into itself: if $\xi \in [2, 3]$ then $T(\xi) = 2 + (\xi - 2)/10 \in [2, 2.1] \subset [2, 3]$.

(A.7) Two-sided absorption. For every $g \geq 2$,

$$e \odot g = 2 \odot g = (2 - 1)^{\ln(g-1)} + 1 = 1^{\ln(g-1)} + 1 = 2 = e,$$

$$g \odot e = g \odot 2 = (g - 1)^{\ln(2-1)} + 1 = (g - 1)^{\ln 1} + 1 = (g - 1)^0 + 1 = 2 = e.$$

Thus $e = 2$ is a two-sided absorbing element for $\odot$, as required by Hypothesis 2 of Theorem 4.1.

(A.8) $k = 2.5$ is a $\odot$ -contractive element. Write $\alpha := \ln(k - 1) = \ln(1.5) \approx 0.4055 \in (0, 1)$.

(C1). Let $a > 2$ and put $A := a - 1 > 1$. Then

$$k \odot a = (k - 1)^{\ln(a-1)} + 1 = (1.5)^{\ln A} + 1 = A^\alpha + 1.$$

Since $A > 1$ and $\alpha < 1$ we have $A^\alpha < A$, hence $k \odot a < A + 1 = a$, which is (C1).

(C2). We first record the closed form of the $\odot$ -powers of k : for every $n \geq 1$,

$$k^{\odot n} = \exp(\alpha^n) + 1. \quad (3)$$

Indeed, for $n = 1$, $\exp(\alpha) + 1 = \exp(\ln 1.5) + 1 = 1.5 + 1 = 2.5 = k$. Assuming (3) for n , the recursion of Remark 2.13 gives

$$\begin{aligned} k^{\odot(n+1)} &= k \odot k^{\odot n} = (k - 1)^{\ln(k^{\odot n} - 1)} + 1 = (1.5)^{\ln(\exp(\alpha^n))} + 1 \\ &= (1.5)^{\alpha^n} + 1 = \exp(\alpha^{n+1}) + 1, \end{aligned}$$

which is (3) for $n + 1$. Now let $a \geq 2$ and $A = a - 1 \geq 1$. Using (3),

$$k^{\odot n} \odot a = (k^{\odot n} - 1)^{\ln(a-1)} + 1 = \exp(\alpha^n)^{\ln A} + 1 = \exp(\alpha^n \ln A) + 1. \quad (4)$$

Since $0 < \alpha < 1$ we have $\alpha^n \rightarrow 0$, hence $\exp(\alpha^n \ln A) \rightarrow \exp(0) = 1$ and therefore $k^{\odot n} \odot a \rightarrow 2 = e$, which is (C2). Thus $k = 2.5$ is $\odot$ -contractive.

(A.9) The pair $(\oplus, \odot)$ satisfies OCP. Let $a \geq 2$, $A = a - 1 \geq 1$, and let $m \geq 1$. Because $x \oplus y = (x - 1)(y - 1) + 1$, an immediate induction shows that for any $x_m, \dots, x_N \in [2, \infty)$,

$$\bigoplus_{i=m}^N x_i = \prod_{i=m}^N (x_i - 1) + 1.$$

Applying this to $x_i = k^{\odot i} \odot a$ and using (4),

$$\bigoplus_{i=m}^N (k^{\odot i} \odot a) = \prod_{i=m}^N \exp(\alpha^i \ln A) + 1 = \exp\left(\ln A \sum_{i=m}^N \alpha^i\right) + 1.$$

Since $0 < \alpha < 1$, the geometric series converges as $N \rightarrow \infty$, giving the finite value

$$\bigoplus_{i=m}^{\infty} (k^{\odot i} \odot a) = \exp\left(\frac{\alpha^m}{1 - \alpha} \ln A\right) + 1 < \infty,$$

so the infinite combination converges in the sense of Definition 2.15. Finally $\alpha^m/(1 - \alpha) \rightarrow 0$ as $m \rightarrow \infty$, whence

$$\bigoplus_{i=m}^{\infty} (k^{\odot i} \odot a) \longrightarrow \exp(0) + 1 = 2 = e,$$

which is exactly OCP. Together with (A.1)–(A.8) and (A.11), this shows that all three hypotheses of Theorem 4.1 hold in Example 4.3.

(A.10) Convergence of the Picard Sequence. For any $x_0 \in [2, 3]$, the Picard sequence generated by the map $T(\xi) = 2 + \frac{\xi - 2}{10}$ of Example 4.3 is

$$x_{n+1} = T(x_n) = 2 + \frac{x_n - 2}{10}.$$

The solution is:

$$x_n = 2 + \frac{x_0 - 2}{10^n}.$$

Indeed:

$$x_{n+1} - 2 = \frac{x_n - 2}{10}.$$

So, by induction:

$$x_n - 2 = \frac{x_0 - 2}{10^n}.$$

Thus:

$$x_n = 2 + \frac{x_0 - 2}{10^n} \longrightarrow 2 \quad \text{as } n \rightarrow \infty.$$

Since $x_0 \in [2, 3]$ implies $x_0 - 2 \in [0, 1]$, the sequence $\{x_n\}$ remains in $[2, 3]$ for all n , and the Picard sequence converges geometrically (with ratio $1/10$) to the fixed point 2, consistent with Theorem 4.1.

(A.11) Associativity of the operation $\odot$. Let

$$x = a - 1, \quad y = b - 1, \quad z = c - 1,$$

where $a, b, c > 1$. Then

$$a \odot b = x^{\ln y} + 1.$$

Therefore,

$$\begin{aligned} (a \odot b) \odot c &= \left(x^{\ln y}\right)^{\ln z} + 1 \\ &= x^{(\ln y)(\ln z)} + 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} a \odot (b \odot c) &= x^{\ln(y^{\ln z})} + 1 \\ &= x^{(\ln z)(\ln y)} + 1 \\ &= x^{(\ln y)(\ln z)} + 1. \end{aligned}$$

Hence,

$$(a \odot b) \odot c = a \odot (b \odot c),$$

showing that $\odot$ is associative on $(1, \infty)$, and in particular on $[2, \infty)$.

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Biography



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